XVII. *Memoir on the Theta-Functions, particularly those of two variables.*

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The following paper is divided into four sections. Section I. deals with what may be called Rosenhain's theory; under the guidance of Professor H. J. S. Smith's paper on the single theta-functions (in vol. i. of London Math. Soc. Proc.), there is investigated a general theorem for the product of four double theta-functions with different characteristics and variables, the definition being

$$\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}=\sum_{m=-\infty}^{m=\infty}\ \sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{1}{4}(2m+\mu)^2}q^{\frac{1}{4}(2n+\nu)^2}r^{\frac{1}{2}(2m+\mu)(2n+\nu)}e^{(2m+\mu)\frac{i\pi x}{2K}+(2n+\nu)\frac{i\pi y}{2\Lambda}},$$

the product being equal to the sum of 16 similar products; and the equation is shown to include 4096 particular cases. Quadratic relations are established between the functions; and the 15 quotients of all of them but one by that one are expressed in terms of two new variables x_1, x_2, the connexion between x_1, x_2 and the original variables x, y being

$$x=\int^{x_1}\frac{A+Bz}{\sqrt{Z}}dz+\int^{x_2}\frac{A+Bz}{\sqrt{Z}}dz$$

$$y=\int^{x_1}\frac{A'+B'z}{\sqrt{Z}}dz+\int^{x_2}\frac{A'+B'z}{\sqrt{Z}}dz$$

where

$$Z=z(1-z)(1-\kappa_1^2z)(1-\kappa_2^2z)(1-\kappa_3^2z)$$

and A, B, A′, B′, κ_1, κ_2, κ_3 are perfectly determinate constants. The quadruple periodicity of the functions is investigated at the beginning of the section, and at the end definite-integral expressions for the periods are obtained.

Section II. gives the expansions of all the functions

(i) in trigonometrical series;
(ii) in ascending powers of x and y.

To obtain the latter, use is made of a theorem there proved :—

$$\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}=e^{-\frac{2K\Lambda\log r}{\pi^2}\frac{d^2}{dxdy}}\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}(y)$$

$\theta_{\mu,\lambda}(x)$, $\theta_{\nu,\rho}(y)$ being single theta-functions. From it are also obtained the expressions for the four periods, as well as a proof of the product theorem of Section I.; and the function Φ is shown to satisfy two differential equations of the form

$$\frac{d^2\Phi}{dx^2}-2x\left(\kappa'^2-\frac{\mathrm{E}}{\mathrm{K}}\right)\frac{d\Phi}{dx}+2\kappa\kappa'^2\frac{d\Phi}{d\kappa}=0$$

(κ, κ', E having the ordinary meaning in reference to $\theta_{\mu,\lambda}(x)$), and an equation of the form

$$r\frac{d\Phi}{dr}+\frac{2\mathrm{K}\Lambda}{\pi^2}\frac{d^2\Phi}{dxdy}=0.$$

Section III. forms the expression of the addition theorem. Although no addition theorem proper exists for theta-functions (that is to say, $\Phi(x+\xi, y+\eta)$ cannot be written down in terms of functions of x, y and of ξ, η), an expression is obtainable for

$$\Phi(x+\xi, y+\eta)\ .\ \Phi'(x-\xi, y-\eta)$$

Φ, Φ' being either the same or different functions. Since any one function of the sum may be combined with any function of the difference of the variables, 256 equations are necessary; and these are written down in 16 sets of 16 each.

In Section IV. many of the properties already proved for the double theta-functions are generalized for the "r" tuple functions. Among these are:—

(i.) The periodicity as in Section I.;

(ii.) The product theorem, which gives the product of four functions as the sum of 4^r products of four functions; from it several general relations are deduced;

(iii.) The analogue of the theorem in Section II., viz.:—

$$\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \mu_1, \mu_2, \ldots, \mu_r\end{pmatrix}x_1, x_2, \ldots, x_r\right\}=e^{-\frac{2}{\pi^2}\sum\limits_{s=1}^{s=r}\sum\limits_{t=1}^{t=r}\mathrm{K}_s\mathrm{K}_t\log p_{s,t}\frac{d^2}{dx_sdx_t}}\prod_{t=1}^{t=r}\theta_{\mu_t,\lambda_t}(x_t)$$

(iv.) The r differential equations of the form

$$\frac{d^2\Phi}{dx_r}-2x_r\left(\kappa'^2_r-\frac{\mathrm{E}_r}{\mathrm{K}_r}\right)\frac{d\Phi}{dx_r}+2\kappa_r\kappa'^2_r\frac{d\Phi}{d\kappa_r}=0$$

and the $\frac{1}{2}r(r-1)$ of the form

$$p_{s,t}\frac{d\Phi}{dp_{s,t}}+\frac{2\mathrm{K}_s\mathrm{K}_t}{\pi^2}\frac{d^2\Phi}{dx_sdx_t}=0$$

all satisfied by Φ.

INTRODUCTION.

Literature of the Subject.

1. The published investigations on the double theta-functions may be conveniently vided into four classes :—

(i.) Those concerning the algebraical integrals of the equations

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$$

$$\frac{xdx}{\sqrt{X}}+\frac{ydy}{\sqrt{Y}}+\frac{zdz}{\sqrt{Z}}=0$$

where X, Y, Z are of the form

$$\xi(1-\xi)(1-\kappa\xi)(1-\lambda\xi)(1-\mu\xi)$$

or

$$(a-\xi)(b-\xi)(c-\xi)(d-\xi)(e-\xi)(f-\xi);$$

(ii.) Those concerning the theta-functions, properly so-called, proceeding from the definitions and investigating the relations which hold between functions of different (and of the same) arguments ;

(iii.) The transformation theory ;

(iv.) The applications to geometry, principally in reference to Kummer's 16-nodal quartic surface.

2. The principal papers are :—

For (i.) I. Abel. His chief memoir is one occurring in the 'Mémoires des Savans Étrangers,' t. vii., 1841 (but presented to the French Academy in 1826), under the title "Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendantes," pp. 176 sqq.; the particular case of (i.) is considered p. 260. Several other papers, less important, on the transcendental functions occur in the collected edition of his works.

II. Jacobi. (α) "Considerationes generales de transcendentibus Abelianis," 'Crelle,' t. ix. (1832), p. 394 ;

(β) "De functionibus quadrupliciter periodicis quibus theoria transcendentium Abelianarum innititur," 'Crelle,' t. xiii. (1835), p. 55 ;

(γ) "Demonstratio nova theorematis Abeliani," 'Crelle,' t. xxiv. (1842), p. 28.

III. Richelot. "Ueber die Integration eines merkwürdigen systems Differentialgleichungen," 'Crelle,' t. xxiii. (1842), p. 354.

IV. CAYLEY. (α) "A Memoir on the Double θ-Functions," 'Crelle,' t. lxxxv. (1878), p. 214;
(β) "On the Double θ-Functions," 'Crelle,' t. lxxxvii. (1879), p. 74;
(γ) "On the Addition of the Double θ-Functions," 'Crelle,' t. lxxxviii. (1880), p. 74.

For (ii.) I. ROSENHAIN. "Mémoire sur les fonctions de deux variables et à quatre périodes," Mém. des Sav. Étr., t. xi., p. 361. This obtained the prize given by the Paris Academy of Sciences in 1846.

II. GÖPEL. "Theoriæ transcendentium Abelianarum primi ordinis adumbratio levis," 'Crelle,' t. xxxv. (1847), p. 277.

III. WEIERSTRASS. "Zur Theorie der Abelschen Functionen," 'Crelle,' t. xlvii. (1854), p. 289; also t. lii. (1856), p. 285.

IV. RIEMANN. "Theorie der Abelschen Functionen," 'Crelle,' t. liv.; Ges. Werke, p. 81.

V. CAYLEY. "A Memoir on the Single and Double Theta-Functions," Phil. Trans., 1881.

VI. BRIOSCHI. "La relazione di GÖPEL per funzioni iperellittiche d'ordine qualunque," Ann. di Mat., t. x. (1881).

For (iii.) I. HERMITE. "Sur la theorie de la transformation des fonctions Abéliennes," Comptes Rendus, t. xl. (1855).

II. KÖNIGSBERGER. (α) "Ueber die Transformation der Abelschen Functionen erster Ordnung," 'Crelle,' t. lxiv., p. 17 (1865). In this occurs part of the addition theorem;
(β) "Ueber die Transformation des zweiten Grades für die Abelschen Functionen erster Ordnung," 'Crelle,' t. lxvii. (1866), p. 58; a continuation of which, dealing with the modular equations, occurs in the Math. Ann., t. i. (1869), p. 163.

For (iv.) I. KUMMER. "Ueber die algebraische Strahlen-systeme," Berl. Abh. (1866).

II. CAYLEY. (α) "On the Double θ-Functions in connexion with a 16-nodal Quartic Surface," 'Crelle,' t. lxxxiii. (1877), p. 210;
(β) "On the 16-nodal Quartic Surface," 'Crelle,' t. lxxxiv. (1878), p. 238.

III. BORCHARDT. "Ueber die Darstellung der Kummerschen Fläche vierter Ordnung mit sechzehn Knotenpunkten durch die Göpelsche biquadratische Relation zwische vier Theta-functionen mit zwei Variabeln," 'Crelle,' t. lxxxiii. (1877), p. 234.

IV. WEBER. "Ueber die Kummersche Fläche vierter Ordnung," 'Crelle,' t. lxxxiv. (1878), p. 332.

Section I.

3. The general double theta-function is defined by the equation

$$\Phi\left\{\begin{pmatrix}\lambda, \rho \\ \mu, \nu\end{pmatrix}x, y\right\}=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{(2m+\mu)^2}{4}}q^{\frac{(2n+\nu)^2}{4}}r^{\frac{(2m+\mu)(2n+\nu)}{2}}v^{x(2m+\mu)}w^{y(2n+\nu)}. \quad . \quad (1)$$

in which λ, μ, ρ, ν are given integers (afterwards taken to be each either zero or unity) and $\begin{pmatrix}\lambda, \rho \\ \mu, \nu\end{pmatrix}$ is called the characteristic; x, y are the variables; p, q, r, v, w are known constants, called the parameters; and the double summation extends to all positive and negative integral values (including zero) of m and n. To ensure the convergence of the doubly infinite series it is necessary that the real part of

$$(2m+\mu)^2\log p+(2n+\nu)^2\log q+2(2m+\mu)(2n+\nu)\log r$$

should be negative for all real values of m and n; beyond this restriction, there is no limitation to the form or the values of p, q and r.

4. It follows at once from the definition that

$$\Phi\left\{\begin{pmatrix}\lambda+2, \rho \\ \mu \quad , \nu\end{pmatrix}\right\}=\Phi\left\{\begin{pmatrix}\lambda, \rho \\ \mu, \nu\end{pmatrix}\right\}=\Phi\left\{\begin{pmatrix}\lambda, \rho+2 \\ \mu, \nu \quad\end{pmatrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$(-1)^\lambda\Phi\left\{\begin{pmatrix}\lambda \quad , \rho \\ \mu+2, \nu\end{pmatrix}\right\}=\Phi\left\{\begin{pmatrix}\lambda, \rho \\ \mu, \nu\end{pmatrix}\right\}=\Phi\left\{\begin{pmatrix}\lambda, \rho \quad \\ \mu, \nu+2\end{pmatrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the variables being the same throughout. Hence there are, in all, sixteen distinct functions, obtained by assigning to the four numbers of the characteristic the values zero and unity and taking all possible combinations.

Also from the definition

$$\Phi\left\{\begin{pmatrix}\lambda, \rho \\ \mu, \nu\end{pmatrix}-x, -y\right\}=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{(2m+\mu)^2}{4}}q^{\frac{(2n+\nu)^2}{4}}r^{\frac{(2m+\mu)(2n+\nu)}{2}}v^{-x(2m+\mu)}w^{-y(2n+\nu)}$$

$$=(-1)^{\lambda\mu+\nu\rho}\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{(m+\mu)\lambda+(n+\nu)\rho}p^{\frac{(-2\overline{m+\mu}+\mu)^2}{4}}q^{\frac{(-2\overline{n+\nu}+\nu)^2}{4}}$$

$$r^{\frac{(-2\overline{m+\mu}+\mu)(-2\overline{n+\nu}+\nu)}{2}}v^{x(-2\overline{m+\mu}+\mu)}w^{y(-2\overline{n+\nu}+\nu)}$$

$$=(-1)^{\lambda\mu+\nu\rho}\sum_{m'=-\infty}^{m'=\infty}\sum_{n'=-\infty}^{n'=\infty}(-1)^{m'\lambda+n'\rho}p^{\frac{(2m'+\mu)^2}{4}}q^{\frac{(2n'+\nu)^2}{4}}r^{\frac{(2m'+\mu)(2n'+\nu)}{2}}v^{x(2m'+\mu)}w^{y(2n'+\nu)}$$

$$=(-1)^{\lambda\mu+\nu\rho}\Phi\left\{\begin{pmatrix}\lambda, \rho \\ \mu, \nu\end{pmatrix}x, y\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence there are ten even and six uneven functions, the latter being denoted by an asterisk in the following table, showing the correspondence between the notations that are used. The current-number notation adopted in this paper was chosen to coincide as closely as possible with that of Professor CAYLEY (with which it is almost identical as will be seen from the table) and with ROSENHAIN'S notation; that of WEIERSTRASS is given as quoted by KÖNIGSBERGER. In the table i denotes $\sqrt{-1}$.

Asterisk denotes odd function. *	HERMITE'S characteristic.	Current-number adopted in this paper.	ROSENHAIN.	GÖPEL.	CAYLEY'S characteristic.	CAYLEY'S current-number.	WEIERSTRASS.
	$\Phi\begin{matrix}0, 0\\0, 0\end{matrix}$	ϑ_0	ϕ_{33}	P'''	$\vartheta\begin{pmatrix}0, 0\\0, 0\end{pmatrix}$	ϑ_0	ϑ_5
	$\begin{matrix}0, 0\\0, 1\end{matrix}$	ϑ_2	ϕ_{32}	Q'''	$\begin{matrix}0, 1\\0, 0\end{matrix}$	ϑ_2	ϑ_{01}
*	$\begin{matrix}0, 1\\0, 1\end{matrix}$	ϑ_{10}	ϕ_{31}	iQ''	$\frac{1}{i}\vartheta\begin{pmatrix}0, 1\\0, 1\end{pmatrix}$	$\frac{1}{i}\vartheta_{10}$	$i\vartheta_{02}$
	$\begin{matrix}0, 1\\0, 0\end{matrix}$	ϑ_8	ϕ_{30}	P''	$\begin{matrix}0, 0\\0, 1\end{matrix}$	ϑ_8	ϑ_{12}
	$\begin{matrix}0, 0\\1, 0\end{matrix}$	ϑ_1	ϕ_{23}	R'''	$\begin{matrix}1, 0\\0, 0\end{matrix}$	ϑ_1	ϑ_4
	$\begin{matrix}0, 0\\1, 1\end{matrix}$	ϑ_3	ϕ_{22}	S'''	$\begin{matrix}1, 1\\0, 0\end{matrix}$	ϑ_3	ϑ_{23}
*	$\begin{matrix}0, 1\\1, 1\end{matrix}$	ϑ_{11}	ϕ_{21}	iS''	$\frac{1}{i}\vartheta\begin{pmatrix}1, 1\\0, 1\end{pmatrix}$	$\frac{1}{i}\vartheta_{11}$	$i\vartheta_{13}$
	$\begin{matrix}0, 1\\1, 0\end{matrix}$	ϑ_9	ϕ_{20}	R''	$\begin{matrix}1, 0\\0, 1\end{matrix}$	ϑ_9	ϑ_{03}
*	$\begin{matrix}1, 0\\1, 0\end{matrix}$	ϑ_5	ϕ_{13}	iR'	$\frac{1}{i}\vartheta\begin{pmatrix}1, 0\\1, 0\end{pmatrix}$	$\frac{1}{i}\vartheta_5$	$i\vartheta_3$
*	$\begin{matrix}1, 0\\1, 1\end{matrix}$	ϑ_7	ϕ_{12}	iS'	$\frac{1}{i}\vartheta\begin{pmatrix}1, 1\\1, 0\end{pmatrix}$	$\frac{1}{i}\vartheta_7$	$i\vartheta_{24}$
	$\begin{matrix}1, 1\\1, 1\end{matrix}$	ϑ_{15}	ϕ_{11}	S	$-\vartheta\begin{pmatrix}1, 1\\1, 1\end{pmatrix}$	$-\vartheta_{15}$	$-\vartheta_{14}$
*	$\begin{matrix}1, 1\\1, 0\end{matrix}$	ϑ_{13}	ϕ_{10}	iR	$\frac{1}{i}\vartheta\begin{pmatrix}1, 0\\1, 1\end{pmatrix}$	$\frac{1}{i}\vartheta_{13}$	$i\vartheta_{04}$
	$\begin{matrix}1\ \ 0\\0, 0\end{matrix}$	ϑ_4	ϕ_{03}	P'	$\begin{matrix}0, 0\\1, 0\end{matrix}$	ϑ_4	ϑ_{34}
	$\begin{matrix}1, 0\\0, 1\end{matrix}$	ϑ_6	ϕ_{02}	Q'	$\begin{matrix}0, 1\\1, 0\end{matrix}$	ϑ_6	ϑ_2
*	$\begin{matrix}1, 1\\0, 1\end{matrix}$	ϑ_{14}	ϕ_{01}	iQ	$\frac{1}{i}\vartheta\begin{pmatrix}0, 1\\1, 1\end{pmatrix}$	$\frac{1}{i}\vartheta_{14}$	$i\vartheta_1$
	$\begin{matrix}1, 1\\0, 0\end{matrix}$	ϑ_{12}	ϕ_{00}	P	$\begin{matrix}0, 0\\1, 1\end{matrix}$	ϑ_{12}	ϑ_0

5. In the general definition of Φ substitute

$$v=e^{\frac{i\pi}{2K}} \quad \ldots \ldots \quad (5)$$

$$w=e^{\frac{i\pi}{2\Lambda}} \quad \ldots \ldots \quad (6)$$

so that

$$\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}e^{\frac{1}{4}\left\{(2m+\mu)^2\log p+(2n+\nu)^2\log q+2(2m+\mu)(2n+\nu)\log r\right\}+\frac{i\pi}{2}\left\{(2m+\mu)\frac{x}{K}+(2n+\nu)\frac{y}{\Lambda}\right\}}$$

Obviously

$$\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}=\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x+4K, y\right\}$$
$$=\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y+4\Lambda\right\} \quad \ldots \ldots \quad (7)$$

so that 4K and zero, zero and 4Λ, form two pairs of actual periods, conjugate in x and y, for Φ.

Since

$$e^{-\frac{\pi^2x^2}{4K^2\log p}}\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}e^{\frac{1}{\log p}\left\{\frac{\pi x}{i2K}+\frac{2m+\mu}{2}\log p\right\}^2+(2n+\nu)\left\{\frac{\pi y}{i2\Lambda}+\frac{2m+\mu}{2}\log r\right\}+\frac{(2n+\nu)^2}{4}\log q}$$

and the right-hand side is unaltered by writing

and
$$\left.\begin{aligned}&x+\frac{4K}{\pi i}\log p \text{ for } x\\ &y+\frac{4\Lambda}{\pi i}\log r \text{ for } y\end{aligned}\right\},$$

$\frac{4K}{\pi i}\log p$, $\frac{4\Lambda}{\pi i}\log r$ form a pair of quasi-periods for Φ, conjugate in x and y. Again

$$e^{-\frac{\pi^2y^2}{4\Lambda^2\log q}}\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}e^{\frac{1}{\log q}\left\{\frac{\pi y}{i2\Lambda}+\frac{2n+\nu}{2}\log q\right\}^2+(2m+\mu)\left\{\frac{\pi x}{i2K}+\frac{2n+\nu}{2}\log r\right\}+\frac{(2m+\mu)^2}{4}\log p}$$

and the right-hand side is unaltered by writing

and
$$\left.\begin{aligned}&x+\frac{4K}{\pi i}\log r \text{ for } x\\ &y+\frac{4\Lambda}{\pi i}\log q \text{ for } y\end{aligned}\right\},$$

so that $\frac{4K}{\pi i}\log r$, $\frac{4\Lambda}{\pi i}\log q$ form another pair of quasi-periods for Φ, conjugate in x and y.

Actual.				Quasi.			
x.	HERMITE'S Notation.	y.	HERMITE'S Notation.	x.	HERMITE'S Notation.	y.	HERMITE'S Notation.
4K	Ω_0	0	Υ_0	$\frac{4K}{\pi i}\log p$	Ω_3	$\frac{4\Lambda}{\pi i}\log r$	Υ_3
0	Ω_1	4Λ	Υ_1	$\frac{4K}{\pi i}\log r$	Ω_4	$\frac{4\Lambda}{\pi i}\log q$	Υ_4

This table exhibits the four pairs of conjugate periods in x and y; one relation between them is immediately deduced, viz. :

$$\Omega_0\Upsilon_3-\Omega_3\Upsilon_0+\Omega_1\Upsilon_2-\Omega_2\Upsilon_1=0$$

an equation which HERMITE makes fundamental in his transformation theory (Comptes Rendus, t. xl.) The following equations, giving the relations for quarter and half period increase of the variables, are easily obtained :—

$$\left.\begin{aligned}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+K,\ y\right\}&=(-1)^{\frac{\mu}{2}}\Phi\left\{\begin{pmatrix}\lambda+1, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\\ \Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+2K,\ y\right\}&=(-1)^{\mu}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\left.\begin{aligned}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y+\Lambda\right\}&=(-1)^{\frac{\nu}{2}}\Phi\left\{\begin{pmatrix}\lambda, & \rho+1\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\\ \Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y+2\Lambda\right\}&=(-1)^{\nu}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{aligned}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{K}{\pi i}\log p,\ y+\frac{\Lambda}{\pi i}\log r\right\}&=p^{-\frac{1}{4}}e^{-\frac{i\pi x}{2K}}\Phi\left\{\begin{pmatrix}\lambda & ,\ \rho\\ \mu+1, & \nu\end{pmatrix}x,\ y\right\}\\ \Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{2K}{\pi i}\log p,\ y+\frac{2\Lambda}{\pi i}\log r\right\}&=p^{-1}e^{-\frac{i\pi x}{K}}(-1)^{\lambda}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\end{aligned}\right\} \quad . \quad . \quad (10)$$

$$\left.\begin{aligned}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{K}{\pi i}\log r,\ y+\frac{\Lambda}{\pi i}\log q\right\}&=q^{-\frac{1}{4}}e^{-\frac{i\pi y}{2\Lambda}}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu+1\end{pmatrix}x,\ y\right\}\\ \Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{2K}{\pi i}\log r,\ y+\frac{2\Lambda}{\pi i}\log q\right\}&=q^{-1}e^{-\frac{i\pi y}{\Lambda}}(-1)^{\rho}\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\end{aligned}\right\} \quad . \quad . \quad (11)$$

$$\left.\begin{aligned}
&\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{K}{\pi i}\log p,\ y+\frac{\Lambda}{\pi i}\log r\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x+\frac{K}{\pi i}\log p,\ y+\frac{\Lambda}{\pi i}\log r\right\}}=\frac{\Phi\left\{\begin{pmatrix}\lambda & , \rho\\ \mu,+1, & \nu\end{pmatrix}x,\ y\right\}}{\Phi\left\{\begin{pmatrix}\lambda' & , \rho'\\ \mu'+1, & \nu'\end{pmatrix}x,\ y\right\}}\\
&\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{2K}{\pi i}\log p,\ y+\frac{2\Lambda}{\pi i}\log r\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x+\frac{2K}{\pi i}\log p,\ y+\frac{2\Lambda}{\pi i}\log r\right\}}=(-1)^{\lambda+\lambda'}\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x,\ y\right\}}
\end{aligned}\right\}\quad . \quad . \quad . \quad (12)$$

$$\left.\begin{aligned}
&\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{K}{\pi i}\log r,\ y+\frac{\Lambda}{\pi i}\log q\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x+\frac{K}{\pi i}\log r,\ y+\frac{\Lambda}{\pi i}\log q\right\}}=\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu+1\end{pmatrix}x,\ y\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'+1\end{pmatrix}x,\ y\right\}}\\
&\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+\frac{2K}{\pi i}\log r,\ y+\frac{2\Lambda}{\pi i}\log q\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x+\frac{2K}{\pi i}\log r,\ y+\frac{2\Lambda}{\pi i}\log q\right\}}=(-1)^{\rho+\rho'}\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x,\ y\right\}}
\end{aligned}\right\}\quad . \quad . \quad . \quad (13)$$

$$\begin{aligned}
&\frac{\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x+s\,K+\frac{tK}{\pi i}\log p+\frac{zK}{\pi i}\log r,\ y+s_1\Lambda+\frac{t\Lambda}{\pi i}\log r+\frac{z\Lambda}{\pi i}\log q\right\}}{\Phi\left\{\begin{pmatrix}\lambda', & \rho'\\ \mu', & \nu'\end{pmatrix}x+s'K+\frac{tK}{\pi i}\log p+\frac{zK}{\pi i}\log r,\ y+s'_1\Lambda+\frac{t\Lambda}{\pi i}\log r+\frac{z\Lambda}{\pi i}\log q\right\}}\\
&\qquad=(-1)^{\frac{1}{2}\{(s+s')\mu+(s_1+s'_1)\nu\}}\frac{\Phi\left\{\begin{pmatrix}\lambda+s, & \rho+s_1\\ \mu+t, & \nu+z\end{pmatrix}x,\ y\right\}}{\Phi\left\{\begin{pmatrix}\lambda'+s', & \rho'+s'_1\\ \mu'+t, & \nu'+z\end{pmatrix}x,\ y\right\}}\quad . \quad (14)
\end{aligned}$$

where, in the last formula, s, s_1, s', s'_1, t, z are integers, and the functions on the right hand side may be reduced by formulæ (2), (3) to functions in which the numbers of the characteristics differ from λ, μ, ρ, ν, λ', μ', ρ', ν', respectively by less than 2. This combination of periods in quotients is similar to the combination of real and imaginary periods in elliptic functions.

The product theorem.

6. We multiply four theta-functions

$$\Phi\left\{\begin{pmatrix}\lambda_1, & \rho_1\\ \mu_1, & \nu_1\end{pmatrix}x_1,\ y_1\right\},\ \Phi\left\{\begin{pmatrix}\lambda_2, & \rho_2\\ \mu_2, & \nu_2\end{pmatrix}x_2,\ y_2\right\},\ \Phi\left\{\begin{pmatrix}\lambda_3, & \rho_3\\ \mu_3, & \nu_3\end{pmatrix}x_3,\ y_3\right\},\ \Phi\left\{\begin{pmatrix}\lambda_4, & \rho_4\\ \mu_4, & \nu_4\end{pmatrix}x_4,\ y_4\right\},$$

such that each sum of the four corresponding numbers in the characteristics is even. Let such a product be denoted by

$$\Pi\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

indicating that there are four functions having numbers λ, μ, ρ, ν with subscript indices 1, 2, 3, 4. Taking the general definition given in (1) for Φ, let

$$M_1+2m_1=M_2+2m_2=M_3+2m_3=M_4+2m_4=m_1+m_2+m_3+m_4$$
$$N_1+2n_1=N_2+2n_2=N_3+2n_3=N_4+2n_4=n_1+n_2+n_3+n_4$$

$$\left.\begin{aligned}2(\Lambda_1+\lambda_1)&=2(\Lambda_2+\lambda_2)=2(\Lambda_3+\lambda_3)=2(\Lambda_4+\lambda_4)=\lambda_1+\lambda_2+\lambda_3+\lambda_4\\2(\sigma_1+\mu_1)&=2(\sigma_2+\mu_2)=2(\sigma_3+\mu_3)=2(\sigma_4+\mu_4)=\mu_1+\mu_2+\mu_3+\mu_4\\2(P_1+\rho_1)&=2(P_2+\rho_2)=2(P_3+\rho_3)=2(P_4+\rho_4)=\rho_1+\rho_2+\rho_3+\rho_4\\2(\sigma'_1+\nu_1)&=2(\sigma'_2+\nu_2)=2(\sigma'_3+\nu_3)=2(\sigma'_4+\nu_4)=\nu_1+\nu_2+\nu_3+\nu_4\end{aligned}\right\} \quad (16)$$

which contain the assumption that $\Sigma\lambda$, $\Sigma\mu$, $\Sigma\nu$, $\Sigma\rho$ are all even; and

$$\left.\begin{aligned}2(X_1+x_1)&=2(X_2+x_2)=2(X_3+x_3)=2(X_4+x_4)=x_1+x_2+x_3+x_4\\2(Y_1+y_1)&=2(Y_2+y_2)=2(Y_3+y_3)=2(Y_4+y_4)=y_1+y_2+y_3+y_4\end{aligned}\right\} \quad . \quad . \quad (17)$$

In the course of the proof the equivalent of the algebraical identities

$$\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+\Lambda_4^2=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\Lambda_1P_1+\Lambda_2P_2+\Lambda_3P_3+\Lambda_4P_4=\lambda_1\rho_1+\lambda_2\rho_2+\lambda_3\rho_3+\lambda_4\rho_4 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

will be used. We have

$$\begin{aligned}4\Pi\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}&=4\Sigma(-1)^{m_1\lambda_1+\ldots+m_4\lambda_4+n_1\rho_1+\ldots+n_4\rho_4}\times p^{\frac{(2m_1+\mu_1)^2+\ldots+(2m_4+\mu_4)^2}{4}}\\&\times v^{(2m_1+\mu_1)x_1+\ldots+(2m_4+\mu_4)x_4}\times q^{\frac{(2n_1+\nu_1)^2+\ldots+(2n_4+\nu_4)^2}{4}}\\&\times w^{(2n_1+\nu_1)y_1+\ldots+(2n_4+\nu_4)y_4}\times r^{\frac{(2m_1+\mu_1)(2n_1+\nu_1)+\ldots+(2m_4+\mu_4)(2n_4+\nu_4)}{2}}\end{aligned} \quad (20)$$

where the summation is over all integral values of the m's and n's from $-\infty$ to $+\infty$. Using (18), (19), the indices on the right-hand side of (20) can be at once transformed; and the following equations give these transformed values:—

$$\begin{aligned}m_1\lambda_1+\ldots+m_4\lambda_4&=\tfrac{1}{2}(M_1\Lambda_1+\ldots+M_4\Lambda_4)\\n_1\rho_1+\ldots+n_4\rho_4&=\tfrac{1}{2}\{N_1P_1+\ldots+N_4P_4\}\\(2m_1+\mu_1)^2+\ldots+(2m_4+\mu_4)^2&=(M_1+\sigma_1)^2+\ldots+(M_4+\sigma_4)^2\\(2n_1+\nu_1)^2+\ldots+(2n_4+\nu_4)^2&=(N_1+\sigma'_1)^2+\ldots+(N_4+\sigma'_4)^2\\(2m_1+\mu_1)x_1+\ldots+(2m_4+\mu_4)x_4&=(M_1+\sigma_1)X_1+\ldots+(M_4+\sigma_4)X_4\\(2n_1+\nu_1)y_1+\ldots+(2n_4+\nu_4)y_4&=(N_1+\sigma'_1)Y_1+\ldots+(N_4+\sigma'_4)Y_4\\(2m_1+\mu_1)(2n_1+\nu_1)+\ldots+(2m_4+\mu_4)(2n_4+\nu_4)&=(M_1+\sigma_1)(N_1+\sigma'_1)+\ldots\\&\quad+(M_4+\sigma_4)(N_4+\sigma'_4)\end{aligned}$$

The substitution of these in (20) gives

$$4\Pi\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}=4\Sigma(-1)^{\frac{1}{2}\{(M_1\Lambda_1+\ \ldots\ +M_4\Lambda_4)+(N_1P_1+\ \ldots\ +N_4P_4)\}}p^{\frac{1}{4}\{(M_1+\sigma_1)^2+\ \ldots\ +(M_4+\sigma_4)^2\}}$$
$$q^{\frac{1}{4}\{(N_1+\sigma'_1)^2+\ \ldots\ +(N_4+\sigma'_4)^2\}}v^{(M_1+\sigma_1)X_1+\ \ldots\ +(M_4+\sigma_4)X_4}w^{(N_1+\sigma'_1)Y_1+\ \ldots\ +(N_4+\sigma'_4)Y_4}$$
$$r^{\frac{1}{2}\{(M_1+\sigma_1)(N_1+\sigma'_1)+\ \ldots\ +(M_4+\sigma_4)(N_4+\sigma'_4)\}} \qquad (21)$$

the summation being taken for all values of the M's and N's defined by the equations, *i.e.*, for all integral values which give integral values to the m's and n's. Now the difference between any two of the M's is even, so that they are either all even, or all uneven. Taking the first of these cases, let

$$M_1=2M'_1,\quad M_2=2M'_2,\quad M_3=2M'_3,\quad M_4=2M'_4$$

then since

$$4m_1=-M_1+M_2+M_3+M_4$$

and similar expressions hold for $4m_2$, $4m_3$, $4m_4$ it is sufficient that

$$M'_1+M'_2+M'_3+M'_4=\text{ even.}$$

Taking the second case, let

$$M_1=2M'_1+1,\quad M_2=2M'_2+1,\quad M_3=2M'_3+1,\quad M_4=2M'_4+1$$

it is sufficient that

$$M'_1+M'_2+M'_3+M'_4=\text{ uneven.}$$

With corresponding quantities substituted for the N's in the two cases exactly similar relations hold. Separate the terms in (21) and denote by

Σ_1.	those in which	$M'_1+\ \ldots\ +M'_4$	= even,	and	$N'_1+\ \ldots\ +N'_4$	= even;
Σ_2.	,,	,, ,,	= even,	,,	,,	= odd;
Σ_3.	,,	,, ,,	= odd,	,,	,,	= even;
Σ_4.	,,	,, ,,	= odd,	,,	,,	= odd.

Also write

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4=2\Lambda'$$
$$\rho_1+\rho_2+\rho_3+\rho_4=P_1+P_2+P_3+P_4=2P';$$

and, for shortness, let Q_1, Q_2, Q_3, Q_4 denote the general terms in Σ_1, Σ_2, Σ_3, Σ_4 respectively, so that

$$(-1)^{M'_1\Lambda_1+\ldots+M'_4\Lambda_4+N_1P_1+\ldots+N'_4P_4}Q_1$$
$$=p^{\frac{1}{4}\{(2M_1+\sigma_1)^2+\ldots\}}q^{\frac{1}{4}\{(2N'_1+\sigma'_1)^2+\ldots\}}r^{\frac{1}{2}\{(2M_1+\sigma'_1)(2N'_1+\sigma'_1)+\ldots\}}v^{(2M'_1+\sigma_1)X_1+\ldots}w^{(2N'_1+\sigma'_1)Y_1+\ldots}$$

$$(-1)^{M_1\Lambda_1+\ldots+M'_4\Lambda_4+N'_1P_1+\ldots+N'_4P_4}Q_2$$
$$=p^{\frac{1}{4}\{(2M'_1+\sigma_1)^2+\ldots\}}q^{\frac{1}{4}\{(2N'_1+1+\sigma'_1)^2+\ldots\}}r^{\frac{1}{2}\{(2M'_1+\sigma_1)(2N'_1+1+\sigma'_1)+\ldots\}}v^{(2M'_1+\sigma_1)X_1+\ldots}w^{(2N'_1+1+\sigma'_1)Y_1+\ldots}$$

$$(-1)^{M_1\Lambda_1+\ldots+M'_4\Lambda_4+N_1P_1+\ldots+N_4P_4}Q_3$$
$$=p^{\frac{1}{4}\{(2M_1+1+\sigma_1)^2+\ldots\}}q^{\frac{1}{4}\{(2N'_1+\sigma_1)^2+\ldots\}}r^{\frac{1}{2}\{(2M'_1+1+\sigma_1)(2N'_1+\sigma_1)+\ldots\}}v^{(2M'_1+1+\sigma_1)X_1+\ldots}w^{(2N_1+\sigma'_1)Y_1+\ldots}$$

$$(-1)^{M_1\Lambda_1+\ldots+M'_4\Lambda_4+N'_1P_1+\ldots+N'_4P_4}Q_4$$
$$=p^{\frac{1}{4}\{(2M'_1+1+\sigma_1)^2+\ldots\}}q^{\frac{1}{4}\{(2N'_1+1+\sigma'_1)^2+\ldots\}}r^{\frac{1}{2}\{(2M_1+1+\sigma_1)(2N_1+1+\sigma'_1)+\ldots\}}v^{(2M'_1+1+\sigma_1)X_1+\ldots}w^{(2N_1+1+\sigma'_1)Y_1+\ldots}$$

and (21) becomes

$$4\Pi\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}=4\Sigma_1.Q_1+(-1)^{P'}4\Sigma_2.Q_2+(-1)^{\Lambda'}4\Sigma_3.Q_3+(-1)^{\Lambda+P'}4\Sigma_4.Q_4\,. \qquad (22)$$

Consider these four sums separately; then

$$4\Sigma_1.Q_1=\Sigma\Sigma Q+\Sigma\Sigma(-1)^{N'_1+N'_2+N'_3+N_4}Q_1+\Sigma\Sigma(-1)^{M'_1+M'_2+M'_3+M'_4}Q_1$$
$$+\Sigma\Sigma(-1)^{N'_1+\ldots+N'_4+M_1+\ldots+M'_4}Q_1$$

where the summations on the right hand side are now taken without restriction for all integral values of the M's and N's between $-\infty$ and $+\infty$; and with similar removal of the restrictions on the values of M and N to which the summation extends

$$4\Sigma_2.Q_2=\Sigma\Sigma Q_2-\Sigma\Sigma(-1)^{N'_1+N'_2+N'_3+N'_4}Q_2+\Sigma\Sigma(-1)^{M'_1+M'_2+M'_3+M'_4}Q_2$$
$$-\Sigma\Sigma(-1)^{N_1+\ldots+N'_4+M'_1+\ldots+M'_4}Q_2$$
$$4\Sigma_3.Q_3=\Sigma\Sigma Q_3+\Sigma\Sigma(-1)^{N'_1+N'_2+N'_3+N'_4}Q_3-\Sigma\Sigma(-1)^{M'_1+M'_2+M'_3+M'_4}Q_3$$
$$-\Sigma\Sigma(-1)^{N'_1+\ldots+N_4+M'_1+\ldots+M'_4}Q_3$$
$$4\Sigma_4.Q_4=\Sigma\Sigma Q_4-\Sigma\Sigma(-1)^{N'_1+N'_2+N'_3+N'_4}Q_4-\Sigma\Sigma(-1)^{M'_1+M'_2+M'_3+M'_4}Q_4$$
$$+\Sigma\Sigma(-1)^{N'_1+\ldots+N'_4+M'_1+\ldots+M'_4}Q_4$$

Thus

$$4\Pi\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}x,\ y\right\}=\text{sum of sixteen double summations.}$$

But each of these double summations is the product of four double theta-functions: thus, in particular,

$Q_1 =$ general term in

$$\Phi\left\{\begin{pmatrix}\Lambda_1, P_1\\ \sigma_1, \sigma'_1\end{pmatrix}X_1, Y_1\right\}\Phi\left\{\begin{pmatrix}\Lambda_2, P_2\\ \sigma_2, \sigma'_2\end{pmatrix}X_2, Y_2\right\}\Phi\left\{\begin{pmatrix}\Lambda_3, P_3\\ \sigma_3, \sigma'_3\end{pmatrix}X_3, Y_3\right\}\Phi\left\{\begin{pmatrix}\Lambda_4, P_4\\ \sigma_4, \sigma'_4\end{pmatrix}X_4, Y_4\right\}$$

$$i.e. \text{ in } \Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma, \sigma'\end{pmatrix}X, Y\right\};$$

$$(-1)^{N_1+N_2+N'_3+N'_4}Q_1 = \text{general term in } \Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma, \sigma'\end{pmatrix}X, Y\right\};$$

and so on for the others; and it follows that the expression of the product theorem is given by

$$4\Pi\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}$$

$$\begin{aligned}
=\ &\Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} + \Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} + \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} + \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma, \sigma'\end{pmatrix}X, Y\right\}\\
&+(-1)^{\Lambda'} \text{ into } \left[\Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} + \Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} - \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} - \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\}\right]\\
&+(-1)^{P'} \text{ into } \left[\Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} - \Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} + \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} - \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\}\right]\\
&+(-1)^{\Lambda'+P'} \text{ into } \left[\Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\} - \Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\} - \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\} + \Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\}\right] \quad (23).
\end{aligned}$$

7. This product theorem does not comprise 16^4 equations, as might be expected; in defining Λ, P, σ, σ' it was assumed that the sums of the four λ's, of the four μ's, of the four ρ's, and of the four ν's, were each even. Thus when λ_1, λ_2, λ_3, are known, λ_4, being limited to the values zero and unity, is also known; and similarly for the other numbers. Hence the number of equations comprised is 16^3, *i.e.*, is 4096.

8. If any uniform increase or decrease be made in a set of corresponding numbers

in the left-hand side; the same increase or decrease occurs in the right-hand side; thus if each of the λ's be increased by unity, by unity also will each of the Λ's be increased. Hence, as immediate deductions from (23), are obtained the following formulæ, including as particular cases, many of ROSENHAIN's formulæ contained in the table at the end of his memoir.

$$\Pi\Phi\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho\\ \mu, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda, \rho+1\\ \mu, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1\\ \mu, \nu\end{pmatrix}$$
$$=\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}\\ \sigma, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}\\ \sigma, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1\\ \sigma, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1\\ \sigma, \sigma'\end{pmatrix} \qquad (24)$$

where on the right-hand side the variables are $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$,
and ,, left ,, ,, ,, $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$.

$$\Pi\Phi\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho\\ \mu, \nu\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda, \rho+1\\ \mu, \nu\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1\\ \mu, \nu\end{pmatrix}$$
$$=(-1)^{\mathrm{P}'}\left[\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}\\ \sigma, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}\\ \sigma, \sigma'+1\end{pmatrix}-\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1\\ \sigma, \sigma'+1\end{pmatrix}-\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1\\ \sigma, \sigma'+1\end{pmatrix}\right] \qquad (25)$$

$$\Pi\Phi\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda+1, \rho\\ \mu, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda, \rho+1\\ \mu, \nu\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1\\ \mu, \nu\end{pmatrix}$$
$$=(-1)^{\Lambda'}\left[\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}\\ \sigma+1, \sigma'\end{pmatrix}-\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}\\ \sigma+1, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1\\ \sigma+1, \sigma'\end{pmatrix}-\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1\\ \sigma+1, \sigma'\end{pmatrix}\right] \qquad (26)$$

$$\Pi\Phi\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda+1, \rho\\ \mu, \nu\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda, \rho+1\\ \mu, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1\\ \mu, \nu\end{pmatrix}$$
$$=(-1)^{\Lambda'+\mathrm{P}'}\left[\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}\\ \sigma+1, \sigma'+1\end{pmatrix}-\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}\\ \sigma+1, \sigma'+1\end{pmatrix}\right.$$
$$\left.-\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1\\ \sigma+1, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1\\ \sigma+1, \sigma'+1\end{pmatrix}\right] \quad . \quad . \quad . \quad . \qquad (27).$$

To these must be added a set of four obtained by increasing each of the x's by $\frac{\mathrm{K}}{\pi i}\log p$ and each of the y's by $\frac{\Lambda}{\pi i}\log r$, conjugate quarter periods: by these substitutions (24) gives

$$\Pi\Phi\begin{pmatrix}\lambda, \rho\\ \mu+1, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho\\ \mu+1, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda, \rho+1\\ \mu+1, \nu\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1\\ \mu+1, \nu\end{pmatrix}$$
$$=\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}\\ \sigma+1, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}\\ \sigma+1, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1\\ \sigma+1, \sigma'\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1\\ \sigma+1, \sigma'\end{pmatrix} \qquad (28).$$

Another set of four is obtained by changing x into $x+\frac{K}{\pi i}\log r$

and y ,, $y+\frac{\Lambda}{\pi i}\log q$:

thus (24) gives

$$\Pi\Phi\begin{pmatrix}\lambda, \rho \\ \mu, \nu+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho \\ \mu, \nu+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda, \rho+1 \\ \mu, \nu+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1 \\ \mu, \nu+1\end{pmatrix}$$
$$=\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P} \\ \sigma, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P} \\ \sigma, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1 \\ \sigma, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1 \\ \sigma, \sigma'+1\end{pmatrix} \quad (29);$$

and a fourth set of four from the same equations by increasing each of the x's and y's by $\frac{K}{\pi i}\log pr$, $\frac{\Lambda}{\pi i}\log rq$ respectively: thus (24) gives

$$\Pi\Phi\begin{pmatrix}\lambda, \rho \\ \mu+1, \nu+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho \\ \mu+1, \nu+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda, \rho+1 \\ \mu+1, \nu+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda+1, \rho+1 \\ \mu+1, \nu+1\end{pmatrix}$$
$$=\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P} \\ \sigma+1, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P} \\ \sigma+1, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda, \mathrm{P}+1 \\ \sigma+1, \sigma'+1\end{pmatrix}+\Pi\Phi\begin{pmatrix}\Lambda+1, \mathrm{P}+1 \\ \sigma+1, \sigma'+1\end{pmatrix} \quad (30).$$

But as might be expected from formulæ (8) and (9) with (17), these equations (28) (29) and (30) could be obtained by uniform increase of the numbers μ and ν in the characteristics of (24).

9. Let the value of a function when both the variables are zero be denoted by c, with the same subscript number as marks the function in the current-number notation; for shortness, let

$$\begin{array}{lcl} \vartheta_r(x, y) & \text{be written} & \vartheta_r \\ \vartheta_r(\xi, \eta) & ,, & \theta_r \\ \text{and } \vartheta_r(x+\xi, y+\eta) & ,, & \vartheta_r(x+\xi). \end{array}$$

Then the following equations are obtained from (23) by assigning suitable values to the numbers of the characteristics and to the variables.

$$\begin{array}{lll}
3c_0^2\,\vartheta_0^2 = & c_1^2\vartheta_1^2+c_2^2\vartheta_2^2+c_3^2\vartheta_3^2+c_4^2\vartheta_4^2+c_6^2\vartheta_6^2+c_8^2\vartheta_8^2+c_9^2\vartheta_9^2+c_{12}^2\vartheta_{12}^2+c_{15}^2\vartheta_{15}^2 & \text{(i)} \\
3c_1^2\,\vartheta_1^2 = & c_0^2\vartheta_0^2+c_2^2\vartheta_2^2+c_3^2\vartheta_3^2-c_4^2\vartheta_4^2-c_6^2\vartheta_6^2+c_8^2\vartheta_8^2+c_9^2\vartheta_9^2-c_{12}^2\vartheta_{12}^2-c_{15}^2\vartheta_{15}^2 & \text{(ii)} \\
3c_2^2\,\vartheta_2^2 = & c_0^2\vartheta_0^2+c_1^2\vartheta_1^2+c_3^2\vartheta_3^2+c_4^2\vartheta_4^2+c_6^2\vartheta_6^2-c_8^2\vartheta_8^2-c_9^2\vartheta_9^2-c_{12}^2\vartheta_{12}^2-c_{15}^2\vartheta_{15}^2 & \text{(iii)} \\
3c_3^2\,\vartheta_3^2 = & c_0^2\vartheta_0^2+c_1^2\vartheta_1^2+c_2^2\vartheta_2^2-c_4^2\vartheta_4^2-c_6^2\vartheta_6^2-c_8^2\vartheta_8^2-c_9^2\vartheta_9^2+c_{12}^2\vartheta_{12}^2+c_{15}^2\vartheta_{15}^2 & \text{(iv)} \\
3c_4^2\,\vartheta_4^2 = & c_0^2\vartheta_0^2-c_1^2\vartheta_1^2+c_2^2\vartheta_2^2-c_3^2\vartheta_3^2+c_6^2\vartheta_6^2+c_8^2\vartheta_8^2-c_9^2\vartheta_9^2+c_{12}^2\vartheta_{12}^2-c_{15}^2\vartheta_{15}^2 & \text{(v)} \\
3c_6^2\,\vartheta_6^2 = & c_0^2\vartheta_0^2-c_1^2\vartheta_1^2+c_2^2\vartheta_2^2-c_3^2\vartheta_3^2+c_4^2\vartheta_4^2-c_8^2\vartheta_8^2+c_9^2\vartheta_9^2-c_{12}^2\vartheta_{12}^2+c_{15}^2\vartheta_{15}^2 & \text{(vi)} \\
3c_8^2\,\vartheta_8^2 = & c_0^2\vartheta_0^2+c_1^2\vartheta_1^2-c_2^2\vartheta_2^2-c_3^2\vartheta_3^2+c_4^2\vartheta_4^2-c_6^2\vartheta_6^2+c_9^2\vartheta_9^2+c_{12}^2\vartheta_{12}^2-c_{15}^2\vartheta_{15}^2 & \text{(vii)} \\
3c_9^2\,\vartheta_9^2 = & c_0^2\vartheta_0^2+c_1^2\vartheta_1^2-c_2^2\vartheta_2^2-c_3^2\vartheta_3^2-c_4^2\vartheta_4^2+c_6^2\vartheta_6^2+c_8^2\vartheta_8^2-c_{12}^2\vartheta_{12}^2+c_{15}^2\vartheta_{15}^2 & \text{(viii)} \\
3c_{12}^2\vartheta_{12}^2 = & c_0^2\vartheta_0^2-c_1^2\vartheta_1^2-c_2^2\vartheta_2^2+c_3^2\vartheta_3^2+c_4^2\vartheta_4^2-c_6^2\vartheta_6^2+c_8^2\vartheta_8^2-c_9^2\vartheta_9^2+c_{15}^2\vartheta_{15}^2 & \text{(ix)} \\
3c_{15}^2\vartheta_{15}^2 = & c_0^2\vartheta_0^2-c_1^2\vartheta_1^2-c_2^2\vartheta_2^2+c_3^2\vartheta_3^2-c_4^2\vartheta_4^2+c_6^2\vartheta_6^2-c_8^2\vartheta_8^2+c_9^2\vartheta_9^2+c_{12}^2\vartheta_{12}^2 & \text{(x)}
\end{array}$$

These are not independent: thus, adding equations (ii)–(x), equation (i) is obtained; and simpler relations equivalent to these will be deduced later on.

$$3c_0^2\vartheta_5^2 = c_4^2\vartheta_1^2+c_6^2\vartheta_3^2-c_1^2\vartheta_4^2-c_3^2\vartheta_6^2-c_2^2\vartheta_7^2+c_{12}^2\vartheta_9^2-c_{15}^2\vartheta_{10}^2-c_9^2\vartheta_{12}^2+c_8^2\vartheta_{13}^2 \quad \text{(xi)}$$

$$3c_2^2\vartheta_7^2 = c_4^2\vartheta_1^2+c_6^2\vartheta_3^2-c_1^2\vartheta_4^2+c_0^2\vartheta_5^2-c_3^2\vartheta_6^2-c_{12}^2\vartheta_9^2+c_{15}^2\vartheta_{10}^2+c_9^2\vartheta_{12}^2-c_8^2\vartheta_{13}^2 \quad \text{(xii)}$$

$$3c_{15}^2\vartheta_{10}^2 = c_4^2\vartheta_1^2-c_6^2\vartheta_3^2-c_1^2\vartheta_4^2-c_0^2\vartheta_5^2+c_3^2\vartheta_6^2+c_2^2\vartheta_7^2-c_{12}^2\vartheta_9^2+c_9^2\vartheta_{12}^2+c_8^2\vartheta_{13}^2 \quad \text{(xiii)}$$

$$3c_8^2\vartheta_{13}^2 = c_4^2\vartheta_1^2-c_6^2\vartheta_3^2-c_1^2\vartheta_4^2+c_0^2\vartheta_5^2+c_3^2\vartheta_6^2-c_2^2\vartheta_7^2+c_{12}^2\vartheta_9^2+c_{15}^2\vartheta_{10}^2-c_9^2\vartheta_{12}^2 \quad \text{(xiv)}$$

$$3c_0^2\vartheta_{13}^2 = c_{12}^2\vartheta_1^2+c_{15}^2\vartheta_2^2-c_9^2\vartheta_4^2+c_8^2\vartheta_5^2+c_4^2\vartheta_9^2-c_6^2\vartheta_{11}^2-c_1^2\vartheta_{12}^2+c_3^2\vartheta_{14}^2-c_2^2\vartheta_{15}^2 \quad \text{(xv)}$$

$$3c_8^2\vartheta_5^2 = c_{12}^2\vartheta_1^2-c_{15}^2\vartheta_2^2-c_9^2\vartheta_4^2+c_4^2\vartheta_9^2+c_6^2\vartheta_{11}^2-c_1^2\vartheta_{12}^2+c_0^2\vartheta_{13}^2-c_3^2\vartheta_{14}^2+c_2^2\vartheta_{15}^2 \quad \text{(xvi)}$$

$$3c_6^2\vartheta_{11}^2 = c_{12}^2\vartheta_1^2-c_{15}^2\vartheta_2^2+c_9^2\vartheta_4^2+c_8^2\vartheta_5^2-c_4^2\vartheta_9^2-c_1^2\vartheta_{12}^2-c_0^2\vartheta_{13}^2+c_3^2\vartheta_{14}^2+c_2^2\vartheta_{15}^2 \quad \text{(xvii)}$$

$$3c_3^2\vartheta_{14}^2 = c_{12}^2\vartheta_1^2+c_{15}^2\vartheta_2^2+c_9^2\vartheta_4^2-c_8^2\vartheta_5^2-c_4^2\vartheta_9^2+c_6^2\vartheta_{11}^2-c_1^2\vartheta_{12}^2+c_0^2\vartheta_{13}^2-c_2^2\vartheta_{15}^2 \quad \text{(xviii)}$$

$$3c_1^2\vartheta_{12}^2 = c_{12}^2\vartheta_1^2+c_{15}^2\vartheta_2^2+c_9^2\vartheta_4^2-c_8^2\vartheta_5^2+c_4^2\vartheta_9^2-c_6^2\vartheta_{11}^2-c_0^2\vartheta_{13}^2-c_3^2\vartheta_{14}^2+c_2^2\vartheta_{15}^2 \quad \text{(xix)}$$

$$3c_{12}^2\vartheta_1^2 = +c_{15}^2\vartheta_2^2+c_9^2\vartheta_4^2+c_8^2\vartheta_5^2+c_4^2\vartheta_9^2+c_6^2\vartheta_{11}^2+c_1^2\vartheta_{12}^2+c_0^2\vartheta_{13}^2+c_3^2\vartheta_{14}^2+c_2^2\vartheta_{15}^2 \quad \text{(xx)}$$

$$3c_2^2\vartheta_{13}^2 = c_{15}^2\vartheta_0^2+c_{12}^2\vartheta_3^2-c_9^2\vartheta_6^2+c_8^2\vartheta_7^2+c_6^2\vartheta_9^2-c_4^2\vartheta_{11}^2-c_3^2\vartheta_{12}^2+c_1^2\vartheta_{14}^2-c_0^2\vartheta_{15}^2 \quad \text{(xxi)}$$

$$3c_3^2\vartheta_{12}^2 = c_{15}^2\vartheta_0^2+c_{12}^2\vartheta_3^2+c_9^2\vartheta_6^2-c_8^2\vartheta_7^2+c_6^2\vartheta_9^2-c_4^2\vartheta_{11}^2-c_2^2\vartheta_{13}^2-c_1^2\vartheta_{14}^2+c_0^2\vartheta_{15}^2 \quad \text{(xxii)}$$

$$3c_{15}^2\vartheta_{14}^2 = -c_1^2\vartheta_0^2+c_0^2\vartheta_1^2+c_3^2\vartheta_2^2-c_2^2\vartheta_3^2-c_4^2\vartheta_5^2+c_6^2\vartheta_7^2+c_9^2\vartheta_8^2-c_8^2\vartheta_9^2+c_{12}^2\vartheta_{13}^2 \quad \text{(xxiii)}$$

$$3c_{12}^2\vartheta_{13}^2 = -c_1^2\vartheta_0^2+c_0^2\vartheta_1^2+c_3^2\vartheta_2^2-c_2^2\vartheta_3^2+c_4^2\vartheta_5^2-c_6^2\vartheta_7^2-c_9^2\vartheta_8^2+c_8^2\vartheta_9^2-c_{15}^2\vartheta_{14}^2 \quad \text{(xxiv)}$$

$$3c_{15}^2\vartheta_{11}^2 = c_4^2\vartheta_0^2-c_6^2\vartheta_2^2-c_0^2\vartheta_4^2-c_1^2\vartheta_5^2+c_2^2\vartheta_6^2+c_3^2\vartheta_7^2-c_{12}^2\vartheta_8^2+c_8^2\vartheta_{12}^2+c_9^2\vartheta_{13}^2 \quad \text{(xxv)}$$

$$3c_9^2\vartheta_{13}^2 = c_4^2\vartheta_0^2-c_6^2\vartheta_2^2-c_0^2\vartheta_4^2+c_1^2\vartheta_5^2+c_2^2\vartheta_6^2-c_3^2\vartheta_7^2+c_{12}^2\vartheta_8^2+c_{15}^2\vartheta_{11}^2-c_8^2\vartheta_{12}^2 \quad \text{(xxvi)}$$

$$c_0 c_2 \vartheta_8 \vartheta_{10} = c_1 c_3 \vartheta_9 \vartheta_{11} + c_4 c_6 \vartheta_{12}\vartheta_{14} \quad \text{(xxvii)}$$

$$\vartheta_0 \vartheta_5 \theta_0 \theta_5 = \vartheta_2\vartheta_7\theta_2\theta_7 + \vartheta_8\vartheta_{13}\theta_8\theta_{13} - \vartheta_{10}\vartheta_{15}\theta_{10}\theta_{15} \quad \text{(xxviii)}$$

$$\vartheta_{10}\vartheta_{11}\theta_{14}\theta_{15} = \vartheta_8\vartheta_9\theta_{12}\theta_{13} + \vartheta_2\vartheta_3\theta_6\theta_7 - \vartheta_0\vartheta_1\theta_4\theta_5 \quad \text{(xxix)}$$

10. By equations (i)–(x) we have

$$c_0^2\vartheta_0^2 - c_{12}^2\vartheta_{12}^2 = c_1^2\vartheta_1^2 + c_6^2\vartheta_6^2 = c_2^2\vartheta_2^2 + c_9^2\vartheta_9^2 \quad (31)$$

$$c_0^2\vartheta_0^2 - c_3^2\vartheta_3^2 = c_6^2\vartheta_6^2 + c_8^2\vartheta_8^2 = c_4^2\vartheta_4^2 + c_9^2\vartheta_9^2 \quad (32)$$

$$c_0^2\vartheta_0^2 - c_{15}^2\vartheta_{15}^2 = c_2^2\vartheta_2^2 + c_8^2\vartheta_8^2 = c_1^2\vartheta_1^2 + c_4^2\vartheta_4^2 \quad (33)$$

giving six distinct expressions for $c_0^2\vartheta_0^2$; and six can be obtained for each of the even functions in this form.

By (xi)–(xiv),

$$-c_8^2\vartheta_{13}^2 + c_{15}^2\vartheta_{10}^2 = c_9^2\vartheta_{12}^2 - c_{12}^2\vartheta_9^2 = -c_0^2\vartheta_5^2 + c_2^2\vartheta_7^2 \quad (34)$$

By (xv)–(xviii),

$$c_0^2\vartheta_{13}^2 - c_8^2\vartheta_5^2 = c_{15}^2\vartheta_2^2 - c_2^2\vartheta_{15}^2 = c_3^2\vartheta_{14}^2 - c_6^2\vartheta_{11}^2 \quad (35)$$

By (xv), (xvii), (xix), (xx),

$$c_1^2\vartheta_{12}^2 + c_0^2\vartheta_{13}^2 = c_4^2\vartheta_9^2 + c_{15}^2\vartheta_2^2 = c_{12}^2\vartheta_1^2 - c_6^2\vartheta_{11}^2 \quad (36)$$

By (xxi), (xxii), and two others,

$$c_2^2\vartheta_{13}^2 + c_3^2\vartheta_{12}^2 = c_6^2\vartheta_9^2 + c_{15}^2\vartheta_0^2 = c_{12}^2\vartheta_3^2 - c_4^2\vartheta_{11}^2 \quad (37)$$

By (xxiii), (xxvi), and two others,

$$c_{15}^2\vartheta_{14}^2-c_{12}^2\vartheta_{13}^2=-c_8^2\vartheta_9^2+c_9^2\vartheta_8^2=-c_4^2\vartheta_5^2+c_6^2\vartheta_7^2 \quad . \quad . \quad . \quad . \quad . \quad (38)$$

By (xxv), (xxvi), and two others,

$$c_{15}^2\vartheta_{11}^2-c_9^2\vartheta_{13}^2=-c_{12}^2\vartheta_8^2+c_8^2\vartheta_{12}^2=-c_1^2\vartheta_5^2+c_3^2\vartheta_7^2 \quad . \quad . \quad . \quad . \quad (39)$$

The first members of (34), (36), (37) are given by ROSENHAIN, in the paper already cited, in his formula (94), and of (38), (39) in his formula (99); but the second members are not noticed. The equivalents of (xxvii), (xxviii), (xxix) are given in his formulæ (98) and (102); and of the following equations (40), and (α)–($\iota\epsilon$), in (89) and (90).

11. By making both the variables zero in (31), (32), (33) there at once follow the equations

$$\left.\begin{aligned} c_0^4-c_{12}^4&=c_1^4+c_6^4=c_2^4+c_9^4\\ c_0^4-c_3^4&=c_6^4+c_8^4=c_4^4+c_9^4\\ c_0^4-c_{15}^4&=c_2^4+c_8^4=c_1^4+c_4^4 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and the following are obtained from (23),

$$\left\{\begin{aligned} c_0^2c_{12}^2&=c_4^2c_8^2+c_3^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\alpha)\\ c_3^2c_{12}^2&=c_6^2c_9^2+c_0^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\beta)\\ c_0^2c_3^2&=c_1^2c_2^2+c_{12}^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\gamma) \end{aligned}\right.$$

$$\left\{\begin{aligned} c_0^2c_4^2&=c_2^2c_6^2+c_8^2c_{12}^2 \quad . \quad . \quad . \quad . \quad (\delta)\\ c_1^2c_4^2&=c_3^2c_6^2+c_9^2c_{12}^2 \quad . \quad . \quad . \quad . \quad (\epsilon)\\ c_0^2c_1^2&=c_2^2c_3^2+c_8^2c_9^2 \quad . \quad . \quad . \quad . \quad (\varsigma) \end{aligned}\right.$$

$$\left\{\begin{aligned} c_0^2c_8^2&=c_1^2c_9^2+c_4^2c_{12}^2 \quad . \quad . \quad . \quad . \quad (\zeta)\\ c_2^2c_8^2&=c_3^2c_9^2+c_6^2c_{12}^2 \quad . \quad . \quad . \quad . \quad (\eta)\\ c_0^2c_2^2&=c_1^2c_3^2+c_4^2c_6^2 \quad . \quad . \quad . \quad . \quad (\theta) \end{aligned}\right.$$

$$\left\{\begin{aligned} c_2^2c_4^2&=c_0^2c_6^2+c_9^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\iota)\\ c_3^2c_4^2&=c_1^2c_6^2+c_8^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\iota\alpha)\\ c_2^2c_{12}^2&=c_6^2c_8^2+c_1^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\iota\beta) \end{aligned}\right.$$

$$\left\{\begin{aligned} c_1^2c_8^2&=c_0^2c_9^2+c_6^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\iota\gamma)\\ c_3^2c_8^2&=c_2^2c_9^2+c_4^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\iota\delta)\\ c_1^2c_{12}^2&=c_4^2c_9^2+c_2^2c_{15}^2 \quad . \quad . \quad . \quad . \quad (\iota\epsilon) \end{aligned}\right.$$

which agree with ROSENHAIN's set except ($\iota\gamma$), in which his left-hand side is equivalent to $c_2^2c_8^2$, probably a misprint. It is worthy of remark that the sum or difference of the subscript numbers is the same for the same equation, which is also the case with many of the equations (31)–(39).

12. Eliminating ϑ_5^2 between (38), (39) we have

$$\vartheta_7^2(c_1^2c_6^2-c_3^2c_4^2)=-c_1^2c_8^2\vartheta_9^2+(c_1^2c_9^2+c_4^2c_{12}^2)\vartheta_8^2-c_4^2c_8^2\vartheta_{12}^2$$

or by ($\iota\alpha$) and (ζ)

$$c_{15}^2\vartheta_7^2=-c_0^2\vartheta_8^2+c_1^2\vartheta_9^2+c_4^2\vartheta_{12}^2$$

Similarly from the same equations

$$c_{15}^2\vartheta_5^2=-c_2^2\vartheta_8^2+c_3^2\vartheta_9^2+c_6^2\vartheta_{12}^2$$

By (35), (36)

$$c_3^2\vartheta_{14}^2=c_{12}^2\vartheta_1^2-c_4^2\vartheta_9^2-c_2^2\vartheta_{15}^2 \qquad (41)$$

$$c_0^2\vartheta_{13}^2=c_{15}^2\vartheta_2^2+c_4^2\vartheta_9^2-c_1^2\vartheta_{12}^2$$

$$c_6^2\vartheta_{11}^2=c_{12}^2\vartheta_1^2-c_{15}^2\vartheta_2^2-c_4^2\vartheta_9^2$$

and therefore by (34)

$$c_0^2\vartheta_{10}^2=c_8^2\vartheta_2^2-c_3^2\vartheta_9^2-c_6^2\vartheta_{12}^2$$

which give the squares of the uneven functions in terms of squares of the even functions.

13. Following ROSENHAIN (91), put

$$\kappa_1^2=\frac{c_1^2c_3^2}{c_0^2c_2^2} \qquad \kappa'^2_1=\frac{c_4^2c_6^2}{c_0^2c_2^2}; \quad \text{hence by } (\theta) \qquad \kappa_1^2+\kappa'^2_1=1 \quad . \quad . \quad . \quad (42)$$

$$\kappa_2^2=\frac{c_3^2c_9^2}{c_2^2c_8^2} \qquad \kappa'^2_2=\frac{c_6^2c_{12}^2}{c_2^2c_8^2}; \quad \text{,,} \quad (\eta) \qquad \kappa_2^2+\kappa'^2_2=1 \quad . \quad . \quad . \quad (43)$$

$$\kappa_3^2=\frac{c_1^2c_9^2}{c_0^2c_8^2} \qquad \kappa'^2_3=\frac{c_4^2c_{12}^2}{c_0^2c_8^2}; \quad \text{,,} \quad (\zeta) \qquad \kappa_3^2+\kappa'^2_3=1 \quad . \quad . \quad . \quad (44).$$

Then

$$\kappa_1^2-\kappa_2^2=\frac{c_3^2}{c_2^2}\,\frac{c_1^2c_8^2-c_0^2c_9^2}{c_0^2c_8^2}=\frac{c_3^2c_6^2c_{15}^2}{c_0^2c_2^2c_8^2}=\mathrm{K}_3^2 \text{ say}; \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\kappa_1^2-\kappa_3^2=\frac{c_1^2c_4^2c_{15}^2}{c_0^2c_2^2c_8^2}=\mathrm{K}_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\kappa_2^2-\kappa_3^2=\frac{c_9^2c_{12}^2c_{15}^2}{c_0^2c_2^2c_8^2}=\mathrm{K}_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and from these the following expressions for the ratios of the c's are easily obtained:

$$\left.\begin{array}{lll} \dfrac{c_1^4}{c_0^4}=\dfrac{\kappa_1^2\kappa_3^2}{\kappa_2^2} & \dfrac{c_4^4}{c_0^4}=\dfrac{\kappa'^2_1\kappa'^2_3}{\kappa'^2_2} & \dfrac{c_9^4}{c_0^4}=\dfrac{\kappa'^2_1\kappa_3^2\mathrm{K}_1^2}{\kappa'^2_2\mathrm{K}_2^2} \\ \dfrac{c_2^4}{c_0^4}=\dfrac{\kappa_3^2\kappa'^2_3\mathrm{K}_3^2}{\kappa_2^2\kappa'^2_2\mathrm{K}_2^2} & \dfrac{c_6^4}{c_0^4}=\dfrac{\kappa'^2_1\kappa_3^2\mathrm{K}_3^2}{\kappa_2^2\mathrm{K}_2^2} & \dfrac{c_{12}^4}{c_0^4}=\dfrac{\kappa_1^2\kappa'^2_3\mathrm{K}_1^2}{\kappa_2^2\mathrm{K}_2^2} \\ \dfrac{c_3^4}{c_0^4}=\dfrac{\kappa_1^2\kappa'^2_3\mathrm{K}_3^2}{\kappa'^2_2\mathrm{K}_2^2} & \dfrac{c_8^4}{c_0^4}=\dfrac{\kappa_1^2\kappa'^2_1\mathrm{K}_1^2}{\kappa_2^2\kappa'^2_2\mathrm{K}_2^2} & \dfrac{c_{15}^4}{c_0^4}=\dfrac{\mathrm{K}_3^2\mathrm{K}_1^2}{\kappa_2^2\kappa'^2_2} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (48).$$

If now these be substituted in the equations of which (41) are a type, a set of algebraical identities is obtained; in fact, putting $\kappa_1^2=a$, $\kappa_2^2=b$, $\kappa_3^2=c$, a, b, c being perfectly independent

$$b(a-c)=a(1-c)(b-c)+ac(a-c)+c(1-a)(a-b)$$
$$(b-c)(a-b)=ac(1-b)+(1-a)(1-c)b-b(1-b)$$

and from these

$$a(1-c)(b-c)(1-b)+c(1-a)(1-b)(a-b)+b(1-c)(1-a)(c-a)$$
$$=(a-b)(b-c)(c-a)$$

and many others similar to these, all of which admit of immediate verification.

14. Two other equations, which will afterwards be useful, are

$$c_8c_9\{\vartheta_{12}(x+\xi)\vartheta_{13}(x-\xi)-\vartheta_{13}(x+\xi)\vartheta_{12}(x-\xi)\}$$
$$=\vartheta_{10}\vartheta_{11}\theta_{14}\theta_{15}-\vartheta_8\vartheta_9\theta_{12}\theta_{13}+\vartheta_2\vartheta_3\theta_6\theta_7-\vartheta_0\vartheta_1\theta_4\theta_5$$
$$=2(\vartheta_2\vartheta_3\theta_6\theta_7-\vartheta_0\vartheta_1\theta_4\theta_5) \text{ by (xxix)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$c_9c_{12}\{\vartheta_9(x+\xi)\vartheta_{12}(x-\xi)-\vartheta_{12}(x+\xi)\vartheta_9(x-\xi)\}$$
$$=\vartheta_0\vartheta_5\theta_0\theta_5-\vartheta_2\vartheta_7\theta_2\theta_7+\vartheta_8\vartheta_{13}\theta_8\theta_{13}-\vartheta_{10}\vartheta_{15}\theta_{10}\theta_{15}$$
$$=2(\vartheta_0\vartheta_5\theta_0\theta_5-\vartheta_2\vartheta_7\theta_2\theta_7) \text{ by (xxviii)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50).$$

Connexion with the hyperelliptic integrals.

15. Taking the fifteen ratios obtained by dividing all the functions but one by that one, it follows from the relations already established as (31), (32), (33), . . . that any thirteen of them can be expressed in terms of the remaining two, or that all these

ratios can be expressed in terms of two new variables. Re-arranging now the first parts of (34), (36), (37), and substituting for the c's in terms of the κ's, we have

$$\left.\begin{aligned} -\frac{\kappa_1}{\kappa_2\kappa_3}\frac{\vartheta_{13}^2}{\vartheta_{12}^2}+\frac{\kappa_1\kappa'_2\kappa'_3}{\kappa'_1\kappa_2\kappa_3}\frac{\vartheta_9^2}{\vartheta_{12}^2}+\frac{\mathrm{K}_2\mathrm{K}_3}{\kappa'_1\kappa_2\kappa_3}\frac{\vartheta_{10}^2}{\vartheta_{12}^2}=1 \\ -\frac{\kappa_2}{\kappa_3\kappa_1}\frac{\vartheta_{13}^2}{\vartheta_{12}^2}+\frac{\kappa'_1\kappa_2\kappa'_3}{\kappa_1\kappa'_2\kappa_3}\frac{\vartheta_9^2}{\vartheta_{12}^2}+\frac{\mathrm{K}_3\mathrm{K}_1}{\kappa_1\kappa'_2\kappa_3}\frac{\vartheta_2^2}{\vartheta_{12}^2}=1 \\ -\frac{\kappa_3}{\kappa_1\kappa_2}\frac{\vartheta_{13}^2}{\vartheta_{12}^2}+\frac{\kappa'_1\kappa'_2\kappa_3}{\kappa_1\kappa_2\kappa'_3}\frac{\vartheta_9^2}{\vartheta_{12}^2}+\frac{\mathrm{K}_1\mathrm{K}_2}{\kappa_1\kappa_2\kappa'_3}\frac{\vartheta_0^2}{\vartheta_{12}^2}=1 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (51)$$

agreeing with ROSENHAIN (95). Assuming x_1, x_2 as the new variables, put

$$\frac{\vartheta_{13}^2}{\vartheta_{12}^2}=\mathrm{A}x_1x_2$$

$$\frac{\vartheta_9^2}{\vartheta_{12}^2}=\mathrm{B}(1-x_1)(1-x_2)$$

then if

$$\frac{\vartheta_{10}^2}{\vartheta_{12}^2}=\mathrm{C}(1-\kappa_1^2x_1)(1-\kappa_1^2x_2)$$

$$\frac{\vartheta_2^2}{\vartheta_{12}^2}=\mathrm{D}(1-\kappa_2^2x_1)(1-\kappa_2^2x_2)$$

$$\frac{\vartheta_0^2}{\vartheta_{12}^2}=\mathrm{E}(1-\kappa_3^2x_1)(1-\kappa_3^2x_2).$$

the equations (51) are satisfied if

$$\mathrm{A}=-\kappa_1\kappa_2\kappa_3 \qquad \mathrm{B}=-\frac{\kappa_1\kappa_2\kappa_3}{\kappa'_1\kappa'_2\kappa'_3} \qquad \mathrm{C}=\frac{\kappa_2\kappa_3}{\kappa'_1\mathrm{K}_3\mathrm{K}_2}$$

$$\mathrm{D}=\frac{\kappa_3\kappa_1}{\kappa'_2\mathrm{K}_1\mathrm{K}_3} \qquad \mathrm{E}=\frac{\kappa_1\kappa_2}{\kappa'_3\mathrm{K}_2\mathrm{K}_1}$$

16. The other ratios involve x_1, x_2 irrationally; thus to find $\left(\frac{\vartheta_8}{\vartheta_{12}}\right)^2$, ROSENHAIN uses equations corresponding to (xxvii), (38), (39), and eliminates ϑ_{11}, ϑ_{14} between them, giving a quadratic in $\left(\frac{\vartheta_8}{\vartheta_{12}}\right)^2$. Having obtained this, the expressions for the other functions follow by substituting for the ratios already found in the equations (31)–(39); and the complete system of expressions for the fifteen ratios is as follows, the $+$ sign being usually taken throughout (see CAYLEY, 'Crelle,' t. 88, p. 81):

$$\frac{\vartheta_{13}^2}{\vartheta_{12}^2} = -\kappa_1\kappa_2\kappa_3 x_1 x_2$$

$$\frac{\vartheta_9^2}{\vartheta_{12}^2} = -\frac{\kappa_1\kappa_2\kappa_3}{\kappa'_1\kappa'_2\kappa'_3}(1-x_1)(1-x_2)$$

$$\frac{\vartheta_{10}^2}{\vartheta_{12}^2} = \frac{\kappa_2\kappa_3}{\kappa'_1 \mathrm{K}_2\mathrm{K}_3}(1-\kappa_1^2 x_1)(1-\kappa_1^2 x_2)$$

$$\frac{\vartheta_2^2}{\vartheta_{12}^2} = \frac{\kappa_3\kappa_1}{\kappa'_2 \mathrm{K}_1\mathrm{K}_3}(1-\kappa_2^2 x_1)(1-\kappa_2^2 x_2)$$

$$\frac{\vartheta_0^2}{\vartheta_{12}^2} = \frac{\kappa_1\kappa_2}{\kappa'_3 \mathrm{K}_1\mathrm{K}_2}(1-\kappa_3^2 x_1)(1-\kappa_3^2 x_2)$$

$$\frac{\vartheta_1^2}{\vartheta_{12}^2} = \frac{\kappa_3}{\kappa'_3 \mathrm{K}_1\mathrm{K}_2} \frac{\left\{\sqrt{x_2(1-x_1)(1-\kappa_1^2x_1)(1-\kappa_2^2x_1)(1-\kappa_3^2x_2)} \pm \sqrt{x_1(1-x_2)(1-\kappa_1^2x_2)(1-\kappa_2^2x_2)(1-\kappa_3^2x_1)}\right\}^2}{(x_1-x_2)^2}$$

$$\frac{\vartheta_3^2}{\vartheta_{12}^2} = \frac{\kappa_2}{\kappa'_2 \mathrm{K}_1\mathrm{K}_3} \frac{\left\{\sqrt{x_2(1-x_1)(1-\kappa_1^2x_1)(1-\kappa_2^2x_2)(1-\kappa_3^2x_1)} \pm \sqrt{x_1(1-x_2)(1-\kappa_1^2x_2)(1-\kappa_2^2x_1)(1-\kappa_3^2x_2)}\right\}^2}{(x_1-x_2)^2}$$

$$\frac{\vartheta_4^2}{\vartheta_{12}^2} = -\frac{\kappa_1\kappa_2}{\kappa'_1\kappa'_2 \mathrm{K}_1\mathrm{K}_2} \frac{\left\{\sqrt{x_1(1-x_1)(1-\kappa_1^2x_2)(1-\kappa_2^2x_2)(1-\kappa_3^2x_1)} \pm \sqrt{x_2(1-x_2)(1-\kappa_1^2x_1)(1-\kappa_2^2x_1)(1-\kappa_3^2x_2)}\right\}^2}{(x_1-x_2)^2}$$

$$\frac{\vartheta_5^2}{\vartheta_{12}^2} = \frac{\kappa_3}{\kappa'_1\kappa'_2 \mathrm{K}_1\mathrm{K}_2} \frac{\left\{\sqrt{x_1(1-x_2)(1-\kappa_1^2x_1)(1-\kappa_2^2x_1)(1-\kappa_3^2x_2)} \pm \sqrt{x_2(1-x_1)(1-\kappa_1^2x_2)(1-\kappa_2^2x_2)(1-\kappa_3^2x_1)}\right\}^2}{(x_1-x_2)^2} \quad (52)$$

$$\frac{\vartheta_6^2}{\vartheta_{12}^2} = -\frac{\kappa_1\kappa_3}{\kappa'_1\kappa'_3 \mathrm{K}_1\mathrm{K}_3} \frac{\left\{\sqrt{x_1(1-x_1)(1-\kappa_1^2x_2)(1-\kappa_2^2x_1)(1-\kappa_3^2x_2)} \pm \sqrt{x_2(1-x_2)(1-\kappa_1^2x_1)(1-\kappa_2^2x_2)(1-\kappa_3^2x_1)}\right\}^2}{(x_1-x_2)^2}$$

$$\frac{\vartheta_7^2}{\vartheta_{12}^2} = \frac{\kappa_2}{\kappa'_1\kappa'_3 \mathrm{K}_1\mathrm{K}_3} \frac{\left\{\sqrt{x_1(1-x_2)(1-\kappa_1^2x_1)(1-\kappa_2^2x_2)(1-\kappa_3^2x_1)} \pm \sqrt{x_2(1-x_1)(1-\kappa_1^2x_2)(1-\kappa_2^2x_1)(1-\kappa_3^2x_2)}\right\}^2}{(x_1-x_2)^2}$$

$$\frac{\vartheta_8^2}{\vartheta_{12}^2} = -\frac{1}{\kappa'_1\kappa'_2\kappa'_3} \frac{\left\{\sqrt{x_2(1-x_2)(1-\kappa_1^2x_1)(1-\kappa_2^2x_1)(1-\kappa_3^2x_1)} \pm \sqrt{x_1(1-x_1)(1-\kappa_1^2x_2)(1-\kappa_2^2x_2)(1-\kappa_3^2x_2)}\right\}^2}{(x_1-x_2)^2}$$

$$\left.\begin{aligned}
\frac{\vartheta_{11}^2}{\vartheta_{12}^2} &= \frac{\kappa_2}{\kappa'_2 K_3 K_1} \\
&\quad \frac{\{\sqrt{x_2(1-x_1)(1-\kappa_1^2 x_2)(1-\kappa_2^2 x_1)(1-\kappa_3^2 x_1)} \pm \sqrt{x_1(1-x_2)(1-\kappa_1^2 x_1)(1-\kappa_2^2 x_2)(1-\kappa_3^2 x_2)}\}^2}{(x_1-x_2)^2} \\
\frac{\vartheta_{14}^2}{\vartheta_{12}^2} &= -\frac{\kappa_2\kappa_3}{\kappa'_2\kappa'_3 K_2 K_3} \\
&\quad \frac{\{\sqrt{x_1(1-x_1)(1-\kappa_1^2 x_1)(1-\kappa_2^2 x_2)(1-\kappa_3^2 x_2)} \pm \sqrt{x_2(1-x_2)(1-\kappa_1^2 x_2)(1-\kappa_2^2 x_1)(1-\kappa_3^2 x_1)}\}^2}{(x_1-x_2)^2} \\
\frac{\vartheta_{15}^2}{\vartheta_{12}^2} &= \frac{\kappa_1}{\kappa'_2\kappa'_3 K_2 K_3} \\
&\quad \frac{\{\sqrt{x_1(1-x_2)(1-\kappa_1^2 x_2)(1-\kappa_2^2 x_1)(1-\kappa_3^2 x_1)} \pm \sqrt{x_2(1-x_1)(1-\kappa_1^2 x_1)(1-\kappa_2^2 x_2)(1-\kappa_3^2 x_2)}\}^2}{(x_1-x_2)^2}
\end{aligned}\right\}$$

which correspond with ROSENHAIN's formula (97).

17. It is now necessary to find relations between x_1, x_2 and x, y. Let

$$c_5 = \frac{d\vartheta_5}{dx_0} \qquad c_7 = \frac{d\vartheta_7}{dx_0}$$

$$c'_5 = \frac{d\vartheta_5}{dy_0} \qquad c'_7 = \frac{d\vartheta_7}{dy_0}$$

where $\frac{d}{dx_0}$, $\frac{d}{dy_0}$ imply that, after the differential of the function has been taken, both the variables are to be put zero. Differentiating the equations (49), (50) with regard to ξ and then putting ξ, η zero, and noticing that

$$\frac{df(x+\xi)}{d\xi} = \frac{df(x+\xi)}{dx}$$

$$\frac{df(x-\xi)}{d\xi} = -\frac{df(x-\xi)}{dx},$$

we have from (49)

$$c_8 c_9 \left(\vartheta_{13}\frac{d\vartheta_{12}}{dx} - \vartheta_{12}\frac{d\vartheta_{13}}{dx}\right) = c_6 c_7 \vartheta_2 \vartheta_3 - c_4 c_5 \vartheta_0 \vartheta_1$$

or

$$c_8 c_9 \frac{d}{dx}\left(\frac{\vartheta_{13}}{\vartheta_{12}}\right) = c_4 c_5 \frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_1}{\vartheta_{12}} - c_6 c_7 \frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_3}{\vartheta_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and similarly from (50)

$$c_9 c_{12} \frac{d}{dx}\left(\frac{\vartheta_9}{\vartheta_{21}}\right) = c_0 c_5 \frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_5}{\vartheta_{12}} - c_2 c_7 \frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

Differentiating the same equations with regard to η and proceeding in the same manner we obtain

$$c_8c_9\frac{d}{dy}\left(\frac{\vartheta_{13}}{\vartheta_{12}}\right)=c_4c'_5\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_1}{\vartheta_{12}}-c_6c'_7\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_3}{\vartheta_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$c_9c_{12}\frac{d}{dy}\left(\frac{\vartheta_9}{\vartheta_{12}}\right)=c_0c'_5\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_5}{\vartheta_{12}}-c_2c'_7\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

which correspond to ROSENHAIN'S (104), (105). Let

$$\left.\begin{aligned} X_1&=x_1(1-x_1)(1-\kappa_1^2x_1)(1-\kappa_2^2x_1)(1-\kappa_3^2x_1)\\ X_2&=x_2(1-x_2)(1-\kappa_1^2x_2)(1-\kappa_2^2x_2)(1-\kappa_3^2x_2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Substituting in (53), (54) from (52),

$$x_1\frac{dx_2}{dx}+x_2\frac{dx_1}{dx}=\frac{\alpha\{\sqrt{X_1}x_2(1-\kappa_3^2x_2)-\sqrt{X_2}x_1(1-\kappa_3^2x_1)\}-\beta\{\sqrt{X_1}x_2(1-\kappa_2^2x_2)-\sqrt{X_2}x_1(1-\kappa_2^2x_1)\}}{x_2-x_1}$$

$$(1-x_1)\frac{dx_2}{dx}+(1-x_2)\frac{dx_1}{dx}$$

$$=\frac{\alpha\{\sqrt{X_1}(1-x_2)(1-\kappa_3^2x_2)-\sqrt{X_2}(1-x_1)(1-\kappa_3^2x_1)\}-\beta\{\sqrt{X_1}(1-x_2)(1-\kappa_2^2x_2)-\sqrt{X_2}(1-x_1)(1-\kappa_2^2x_1)\}}{x_2-x_1}$$

where α, β are functions of κ_1, κ_2, κ_3 and of c_5, c_7 which will be afterwards seen to be themselves functions of κ_1, κ_2, κ_3. From these

$$\frac{dx_2}{dx}=\frac{\beta(1-\kappa_2^2x_1)-\alpha(1-\kappa_3^2x_1)}{x_2-x_1}\sqrt{X_2}=\frac{\gamma+\delta x_1}{x_2-x_1}\sqrt{X_2}, \text{ say;}$$

$$\frac{dx_1}{dx}=\frac{\alpha(1-\kappa_3^2x_2)-\beta(1-\kappa_2^2x_2)}{x_2-x_1}\sqrt{X_1}=-\frac{\gamma+\delta x_2}{x_2-x_1}\sqrt{X_1}.$$

Similarly

$$\frac{dx_2}{dy}=-\frac{\gamma'+\delta'x_1}{x_2-x_1}\sqrt{X_2}$$

$$\frac{dx_1}{dy}=\frac{\gamma'+\delta'x_2}{x_2-x_1}\sqrt{X_1}.$$

If, then, we write

$$\left.\begin{aligned} dx&=\frac{A+Bx_1}{\sqrt{X_1}}\,dx_1+\frac{A+Bx_2}{\sqrt{X_2}}\,dx_2\\ dy&=\frac{A'+B'x_1}{\sqrt{X_1}}dx_1+\frac{A'+B'x_2}{\sqrt{X_2}}dx_2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

the foregoing equations are satisfied, provided

$$\begin{aligned}
B\gamma - A\delta &= 1 \\
A'\delta' - B'\gamma' &= 1 \\
B'\gamma - A'\delta &= 0 \\
B\gamma' - A\delta' &= 0
\end{aligned}$$

hence

$$\frac{B}{\delta'} = \frac{A}{\gamma'} = \frac{1}{\gamma\delta' - \gamma'\delta}$$

$$\frac{B'}{\delta} = \frac{A'}{\gamma} = \frac{1}{\gamma\delta' - \gamma'\delta}$$

and therefore A, B, A′, B′ are determinate functions of κ_1, κ_2, κ_3. The equations (58) are the well-known equations for the hyperelliptic integrals of the first kind.

On the expressions of the quarter-periods as definite integrals.

18. Integrating (58)

$$x = \int_0^{x_1} \frac{A + Bx}{\sqrt{X}} dx + \int_a^{x_2} \frac{A + Bx}{\sqrt{X}} dx$$

$$y = \int_0^{x_1} \frac{A' + B'x}{\sqrt{X}} dx + \int_b^{x_2} \frac{A' + B'x}{\sqrt{X}} dx$$

a, b being constants: we proceed to find some integrals giving the values of the periods.

19. (i.) Let $x=0$, $y=0$, so that all the uneven functions vanish; then, by (52), $x_1=0$, $x_2=\frac{1}{\kappa_1^2}$; and hence

$$0 = \int_a^{\frac{1}{\kappa_1^2}} \frac{A + Bx}{\sqrt{X}} dx$$

$$0 = \int_b^{\frac{1}{\kappa^2}} \frac{A' + B'x}{\sqrt{X}} dx.$$

(ii.) Let $x=K$, $y=0$; referring to formula (8) the functions which vanish are seen to be ϑ_1, ϑ_3, ϑ_9, ϑ_{10}, ϑ_{14}, ϑ_{15}; and to ensure this $x_1=1$, $x_2=\frac{1}{\kappa_1^2}$, so that

$$K = \int_0^1 \frac{A + Bx}{\sqrt{X}} dx + \int_a^{\frac{1}{\kappa_1^2}} \frac{A + Bx}{\sqrt{X}} dx$$

$$0 = \int_0^1 \frac{A' + B'x}{\sqrt{X}} dx + \int_b^{\frac{1}{\kappa_1^2}} \frac{A' + B'x}{\sqrt{X}} dx.$$

(iii.) Let $x=0$, $y=\Lambda$: the vanishing functions are ϑ_2, ϑ_3, ϑ_5, ϑ_6, ϑ_{13}, ϑ_{15}; hence $x_1=\frac{1}{\kappa_2^2}$, $x_2=0$, and

$$0=\int_0^{\frac{1}{\kappa_2^2}}\frac{A+Bx}{\sqrt{X}}dx+\int_a^0\frac{A+Bx}{\sqrt{X}}dx$$

$$\Lambda=\int_0^{\frac{1}{\kappa_2^2}}\frac{A'+B'x}{\sqrt{X}}dx+\int_b^0\frac{A'+B'x}{\sqrt{X}}dx.$$

(iv.) Let $x=\frac{K}{\pi i}\log pr$, $y=\frac{\Lambda}{\pi i}\log rq$: the vanishing functions are ϑ_4, ϑ_6, ϑ_8, ϑ_9, ϑ_{13}, ϑ_{14}; hence $x_1=0$, $x_2=1$, and

$$\frac{K}{\pi i}\log pr=\int_a^1\frac{A+Bx}{\sqrt{X}}dx$$

$$\frac{\Lambda}{\pi i}\log rq=\int_b^1\frac{A'+B'x}{\sqrt{X}}dx.$$

(v.) Let $x=K+\frac{K}{\pi i}\log r$, $y=\Lambda+\frac{\Lambda}{\pi i}\log q$: the vanishing functions are ϑ_0, ϑ_3, ϑ_4, ϑ_5, ϑ_9, ϑ_{11}; hence $x_1=\frac{1}{\kappa_3^2}$, $x_2=1$, and

$$K+\frac{K}{\pi i}\log r=\int_0^{\frac{1}{\kappa_3^2}}\frac{A+Bx}{\sqrt{X}}dx+\int_a^1\frac{A+Bx}{\sqrt{X}}dx$$

$$\Lambda+\frac{\Lambda}{\pi i}\log q=\int_0^{\frac{1}{\kappa_3^2}}\frac{A'+B'x}{\sqrt{X}}dx+\int_b^1\frac{A'+B'x}{\sqrt{X}}dx.$$

By the elimination of a, b these ten equations reduce to the following eight :—

$$\left.\begin{aligned}0&=\int_0^1\frac{A'+B'x}{\sqrt{X}}dx\\ K&=\int_0^1\frac{A+Bx}{\sqrt{X}}dx\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(59)$$

$$\left.\begin{aligned}\frac{Ki}{\pi}\log pr&=\int_1^{\frac{1}{\kappa_1^2}}\frac{A+Bx}{\sqrt{X}}dx\\ \frac{\Lambda i}{\pi}\log rq&=\int_1^{\frac{1}{\kappa_1^2}}\frac{A'+B'x}{\sqrt{X}}dx\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(60)$$

$$\left.\begin{aligned}0&=\int_{\frac{1}{\kappa_1^2}}^{\frac{1}{\kappa_2^2}}\frac{A+Bx}{\sqrt{X}}dx\\ \Lambda&=\int_{\frac{1}{\kappa_1^2}}^{\frac{1}{\kappa_2^2}}\frac{A'+B'x}{\sqrt{X}}dx\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(61)$$

$$\left.\begin{aligned}\frac{K}{\pi i}\log r&=\int_{\frac{1}{\kappa_2^2}}^{\frac{1}{\kappa_3^2}}\frac{A+Bx}{\sqrt{X}}dx\\ \frac{\Lambda}{\pi i}\log q&=\int_{\frac{1}{\kappa_2^2}}^{\frac{1}{\kappa_3^2}}\frac{A'+B'x}{\sqrt{X}}dx\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (62).$$

20. Let

$$\left.\begin{aligned}\int_0^1\frac{dx}{\sqrt{X}}&=K_{01} & \int_0^1\frac{xdx}{\sqrt{X}}&=K_{11}\\ \int_0^1\frac{x^2dx}{\sqrt{X}}&=K_{21} & \int_0^1\frac{x^3dx}{\sqrt{X}}&=K_{31}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (63)$$

Then

$$\frac{1}{\kappa_1}\frac{dK_{01}}{d\kappa_1}-\kappa_1\frac{dK_{11}}{d\kappa_1}=K_{11}$$

$$\frac{1}{\kappa_1}\frac{dK_{11}}{d\kappa_1}-\kappa_1\frac{dK_{21}}{d\kappa_1}=K_{21}$$

$$\frac{1}{\kappa_1}\frac{dK_{21}}{d\kappa_1}-\kappa_1\frac{dK_{31}}{d\kappa_1}=K_{31}$$

And

$$\kappa_1\frac{dK_{01}}{d\kappa_1}+K_{01}=\int_0^1\frac{dx}{\sqrt{X}(1-\kappa_1^2x)}$$

and

$$\frac{d}{dx}\cdot\frac{\sqrt{X}}{1-\kappa_1^2x}=\frac{P_1}{\sqrt{X}.(1-\kappa_1^2x)}-\frac{R_1+S_1x+T_1x^2+U_1x^2}{\sqrt{X}}$$

so that

$$\kappa_1\frac{dK_{01}}{d\kappa_1}+K_{01}=PK_{31}+QK_{21}+RK_{11}+SK_{01}$$

P, Q, R, S being determinate functions of κ_1, κ_2, κ_3. By eliminating K_{11}, K_{21}, K_{31} from these equations it is obvious that K_{01} satisfies a differential equation of the fourth order in κ_1 as the independent variable. K_{11} will satisfy a similar equation; and hence also K, equal to

$$AK_{01}+BK_{11},$$

(A, B being functions of κ_1, κ_2, κ_3,) will satisfy a linear differential equation in κ_1 of the fourth order.

Section II.

21. If $\theta_{\mu,\lambda}$ be the general single theta-function, then

$$\theta_{\mu,\lambda}(x)=\sum_{m=-\infty}^{m=\infty}(-1)^{m\lambda}p^{\frac{(2m+\mu)^2}{4}}e^{(2m+\mu)\frac{i\pi x}{2K}}$$

[This notation, since it is already in use, is adopted in preference to $\theta\left\{\binom{\lambda}{\mu}x\right\}$ which would better agree with the definition of the double theta-function in (1) and with that of the "r" tuple function to be given later.]

As is well known, there are three even functions and uneven ; these are

(even) $$\theta_{0,0}(x)=1+2p\cos 2x\frac{\pi}{2K}+2p^4\cos 4x\frac{\pi}{2K}+2p^9\cos 6x\frac{\pi}{2K}+\ldots$$

(even) $$\theta_{0,1}(x)=1-2p\cos 2x\frac{\pi}{2K}+2p^4\cos 4x\frac{\pi}{2K}-2p^9\cos 6x\frac{\pi}{2K}+\ldots$$

(even) $$\theta_{1,0}(x)=2p^{\frac{1}{4}}\cos x\frac{\pi}{2K}+2p^{\frac{9}{4}}\cos 3x\frac{\pi}{2K}+2p^{\frac{25}{4}}\cos 5x\frac{\pi}{2K}+\ldots$$

(uneven) $$\frac{1}{i}\theta_{1,1}(x)=2p^{\frac{1}{4}}\sin x\frac{\pi}{2K}-2p^{\frac{9}{4}}\sin 3x\frac{\pi}{2K}+2p^{\frac{25}{4}}\sin 5x\frac{\pi}{2K}+\ldots$$

and

$$sn\,x=\frac{1}{i\sqrt{\kappa}}\frac{\theta_{1,1}(x)}{\theta_{0,1}(x)}$$

$$cn\,x=\sqrt{\frac{\kappa'}{\kappa}}\frac{\theta_{1,0}(x)}{\theta_{0,1}(x)}$$

$$dn\,x=\sqrt{\kappa'}\frac{\theta_{0,0}(x)}{\theta_{0,1}(x)}$$

22. Writing in the definition of Φ given in (1)

$$v=e^{\frac{i\pi}{2K}} \qquad w=e^{\frac{i\eta}{2\Lambda}}$$

we have the following series of expressions for the ϑ's.

$$\vartheta_0=1+2\sum_{m=1}^{m=\infty}p^{m^2}\cos\frac{m\pi x}{K}+2\sum_{m=1}^{m=\infty}q^{n^2}\cos\frac{n\pi y}{\Lambda}$$
$$+2\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}p^{m^2}q^{n^2}\left\{r^{2mn}\cos\pi\left(\frac{mx}{K}+\frac{ny}{\Lambda}\right)+r^{-2mn}\cos\pi\left(\frac{mx}{K}-\frac{ny}{\Lambda}\right)\right\}\quad . \quad . \quad (64)$$

and therefore

$$2\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} p^{m^2}q^{n^2}\left\{r^{2mn}\cos\pi\left(\frac{mx}{K}+\frac{ny}{\Lambda}\right)+r^{-2mn}\cos\pi\left(\frac{mx}{K}-\frac{ny}{\Lambda}\right)\right\}=\vartheta_0-\theta_{0,0}(x)-\theta_{0,0}(y)+1 \quad . \quad (65)$$

the parameters of $\theta_{0,0}(x)$ being p and K, and of $\theta_{0,0}(y)$ q and Λ, the same applying to the other single theta-functions of x and of y which occur below.

$$\vartheta_1=2\sum_{m=0}^{m=\infty} p^{(m+\frac{1}{2})^2}\cos(m+\tfrac{1}{2})\frac{\pi x}{K}+2\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty} p^{(m+\frac{1}{2})^2}q^{n^2}\left[r^{(2m+1)n}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+\frac{ny}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+1)n}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-\frac{ny}{\Lambda}\right\}\right] \quad (66);$$

and hence

$$2\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty} p^{(m+\frac{1}{2})^2}q^{n^2}\left[r^{(2m+1)n}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+\frac{ny}{\Lambda}\right\}+r^{-(2m+1)n}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-\frac{ny}{\Lambda}\right\}\right]$$
$$=\vartheta_1-\theta_{1,0}(x) \quad (67).$$

$$\vartheta_2=2\sum_{n=0}^{n=\infty} q^{(n+\frac{1}{2})^2}\cos)\,(n+\tfrac{1}{2})\frac{\pi y}{\Lambda}+2\sum_{n=0}^{n=\infty}\sum_{m=1}^{m=\infty} p^{m^2}q^{(n+\frac{1}{2})^2}\left[r^{(2n+1)m}\cos\pi\left\{\frac{mx}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2n+1)m}\cos\pi\left\{\frac{mx}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]. \quad (68);$$

and hence

$$2\sum_{n=0}^{n=\infty}\sum_{m=1}^{m=\infty} p^{m^2}q^{(n+\frac{1}{2})^2}\left[r^{(2n+1)m}\cos\pi\left\{\frac{mx}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}+r^{-(2n+1)m}\cos\pi\left\{\frac{mx}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]$$
$$=\vartheta_2-\theta_{1,0}(y) \quad (69).$$

$$\vartheta_3=2\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} p^{(m+\frac{1}{2})^2}q^{(n+\frac{1}{2})^2}\left[r^{2(m+\frac{1}{2})(n+\frac{1}{2})}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-2(m+\frac{1}{2})(n+\frac{1}{2})}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right] \quad . \quad . \quad (70).$$

Similarly

$$2\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^m p^{m^2}q^{n^2}\left[r^{2mn}\cos\pi\left(\frac{mx}{K}+\frac{ny}{\Lambda}\right)+r^{-2mn}\cos\pi\left(\frac{mx}{K}-\frac{ny}{\Lambda}\right)\right]$$
$$=\vartheta_4-\theta_{0,1}(x)-\theta_{0,0}(y)+1 \quad . \quad . \quad . \quad (71).$$

$$2\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^{m+\frac{1}{2}}p^{(m+\frac{1}{2})^2}q^{n^2}\left[r^{(2m+1)n}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+\frac{ny}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+1)}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-\frac{ny}{\Lambda}\right\}\right]=\vartheta_5-\theta_{1,1}(x) \quad . \quad . \quad . \quad (72).$$

$$2\sum_{m=1}^{m=\infty}\sum_{n=0}^{n=\infty}(-1)^m p^{m^2} q^{(n+\frac{1}{2})^2}\left[r^{(2n+1)m}\cos\pi\left\{\frac{mx}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2n+1)m}\cos\pi\left\{\frac{mx}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_6-\theta_{1,0}(y) \quad . \quad . \quad . \quad (73).$$

$$2\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}(-1)^{m+\frac{1}{2}} p^{(m+\frac{1}{2})^2} q^{(n+\frac{1}{2})^2}\left[r^{2(m+\frac{1}{2})(n+\frac{1}{2})}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+\frac{1}{2})(n+\frac{1}{2})}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_7 \quad . \quad . \quad . \quad (74).$$

$$2\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^n p^{m^2} q^{n^2}\left[r^{2mn}\cos\pi\left(\frac{mx}{K}+\frac{ny}{\Lambda}\right)+r^{-2mn}\cos\pi\left(\frac{mx}{K}-\frac{ny}{\Lambda}\right)\right]$$
$$=\vartheta_8-\theta_{0,0}(x)-\theta_{0,1}(y)+1 \quad . \quad . \quad (75).$$

$$2\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^n p^{(m+\frac{1}{2})^2} q^{n^2}\left[r^{(2m+1)n}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+\frac{ny}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+1)n}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-\frac{ny}{\Lambda}\right\}\right]=\vartheta_9-\theta_{1,0}(x) \quad . \quad . \quad (76).$$

$$2\sum_{m=1}^{m=\infty}\sum_{n=0}^{n=\infty}(-1)^{n+\frac{1}{2}} p^{m^2} q^{(n+\frac{1}{2})^2}\left[r^{(2n+1)m}\sin\pi\left\{\frac{mx}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2n+1)m}\sin\pi\left\{\frac{mx}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_{10}-\theta_{1,1}(y) \quad . \quad . \quad . \quad (77).$$

$$2\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}(-1)^{n+\frac{1}{2}} p^{(m+\frac{1}{2})^2} q^{(n+\frac{1}{2})^2}\left[r^{2(m+\frac{1}{2})(n+\frac{1}{2})}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-2(m+\frac{1}{2})(n+\frac{1}{2})}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_{11} \quad . \quad . \quad (78).$$

$$2\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^{m+n} p^{m^2} q^{n^2}\left[r^{2mn}\cos\pi\left(\frac{mx}{K}+\frac{ny}{\Lambda}\right)+r^{-2mn}\cos\pi\left(\frac{mx}{K}+\frac{ny}{\Lambda}\right)\right]$$
$$=\vartheta_{12}-\theta_{0,1}(x)-\theta_{0,1}(y)+1 \quad . \quad . \quad . \quad (79).$$

$$2\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^{m+\frac{1}{2}+n} p^{(m+\frac{1}{2})^2} q^{n^2}\left[r^{(2m+1)n}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}+\frac{ny}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+1)n}\sin\pi\left\{(m+\tfrac{1}{2})\frac{x}{K}-\frac{ny}{\Lambda}\right\}\right]=\vartheta_{13}-\theta_{1,1}(x) \quad . \quad . \quad . \quad (80).$$

$$2\sum_{m=1}^{m=\infty}\sum_{n=0}^{n=\infty}(-1)^{m+n+\frac{1}{2}} p^{m^2} q^{(n+\frac{1}{2})^2}\left[r^{(2n+1)m}\sin\pi\left\{\frac{mx}{K}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2n+1)m}\sin\pi\left\{\frac{mx}{K}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_{14}-\theta_{1,1}(y) \quad . \quad . \quad . \quad (81).$$

$$2\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}(-1)^{m+n}p^{(m+\frac{1}{2})^2}q^{(n+\frac{1}{2})^2}\left[r^{2(m+\frac{1}{2})(n+\frac{1}{2})}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{\mathrm{K}}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.-r^{-2(m+\frac{1}{2})(n+\frac{1}{2})}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{\mathrm{K}}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_{15}\quad . \quad . \quad (82).$$

From these formulæ many other double summations may be deduced: the following may be taken as specimens.

$$8\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}p^{4m^2}q^{4n^2}\left[r^{8mn}\cos 2\pi\left(\frac{mx}{\mathrm{K}}+\frac{ny}{\Lambda}\right)+r^{-8mn}\cos 2\pi\left(\frac{mx}{\mathrm{K}}-\frac{ny}{\Lambda}\right)\right]$$
$$=\vartheta_0+\vartheta_4+\vartheta_8+\vartheta_{12}-2\theta_{0,0}(x)-2\theta_{0,0}(y)-2\theta_{0,1}(x)-2\theta_{0,1}(y)+4\quad . \quad (83).$$

$$8\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}p^{(2m+1)^2}q^{4n^2}\left[r^{4n(2m+1)}\cos\pi\left\{\frac{(2m+1)x}{\mathrm{K}}+\frac{2ny}{\Lambda}\right\}+r^{-4n(2m+1)}\cos\pi\left\{\frac{(2m+1)x}{\mathrm{K}}-\frac{2ny}{\Lambda}\right\}\right]$$
$$=\vartheta_0-\vartheta_4+\vartheta_8-\vartheta_{12}-2\theta_{0,0}(x)+2\theta_{0,1}(x)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84).$$

$$8\sum_{m=1}^{m=\infty}\sum_{n=0}^{n=\infty}p^{4m^2}q^{(2n+1)^2}\left[r^{4m(2n+1)}\cos\pi\left\{\frac{2mx}{\mathrm{K}}+\frac{(2n+1)y}{\Lambda}\right\}+r^{-4m(2n+1)}\cos\pi\left\{\frac{2mx}{\mathrm{K}}-\frac{(2n+1)y}{\Lambda}\right\}\right]$$
$$=\vartheta_0+\vartheta_4-\vartheta_8-\vartheta_{12}-2\theta_{0,0}(y)+2\theta_{0,1}(y)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85).$$

$$8\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}p^{(2m+1)^2}q^{(2n+1)^2}\left[r^{2(2m+1)(2n+1)}\cos\pi\left\{\frac{(2m+1)x}{\mathrm{K}}+\frac{(2n+1)y}{\Lambda}\right\}\right.$$
$$\left.+r^{-2(2m+1)(2n+1)}\cos\pi\left\{\frac{(2m+1)x}{\mathrm{K}}-\frac{(2n+1)y}{\Lambda}\right\}\right]=\vartheta_0-\vartheta_4-\vartheta_8+\vartheta_{12}\quad . \quad (86).$$

$$4\sum_{m=1}^{m=\infty}\sum_{n=0}^{n=\infty}p^{4m^2}q^{(n+\frac{1}{2})^2}\left[r^{2m(2n+1)}\cos\pi\left\{\frac{2mx}{\mathrm{K}}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}+r^{-2m(2n+1)}\cos\pi\left\{\frac{2mx}{\mathrm{K}}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]$$
$$=\vartheta_2+\vartheta_6-2\theta_{1,0}(y)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87).$$

$$4\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}p^{(2m+1)^2}q^{(n+\frac{1}{2})^2}\left[r^{(2m+1)(2n+1)}\cos\pi\left\{\frac{(2m+1)x}{\mathrm{K}}+(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+1)(2n+1)}\cos\pi\left\{\frac{(2m+1)x}{\mathrm{K}}-(n+\tfrac{1}{2})\frac{y}{\Lambda}\right\}\right]=\vartheta_2-\vartheta_6\quad . \quad (88).$$

$$4\sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}p^{(m+\frac{1}{2})^2}q^{4n^2}\left[r^{2n(2m+1)}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{\mathrm{K}}+\frac{2ny}{\Lambda}\right\}+r^{-2n(2m+1)}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{\mathrm{K}}-\frac{2ny}{\Lambda}\right\}\right]$$
$$=\vartheta_1+\vartheta_9-2\theta_{1,0}(x)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89).$$

$$4\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}p^{(m+\frac{1}{2})^2}q^{(2n+1)^2}\left[r^{(2m+1)(2n+1)}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{\mathrm{K}}+\frac{(2n+1)y}{\Lambda}\right\}\right.$$
$$\left.+r^{-(2m+1)(2n+1)}\cos\pi\left\{(m+\tfrac{1}{2})\frac{x}{\mathrm{K}}-\frac{(2n+1)y}{\Lambda}\right\}\right]=\vartheta_1-\vartheta_9\quad . \quad (90).$$

24. Writing $\log r = \frac{i\pi}{2\mathrm{K}} \log \rho'$, then

$$\Phi\left\{\begin{pmatrix}\lambda, & \rho \\ \mu, & \nu\end{pmatrix} x, y\right\} = \sum_{m=-\infty}^{m=\infty} \sum_{m=-\infty}^{n=\infty} (-1)^{m\lambda+n\rho} p^{\frac{(2m+\mu)^2}{4}} q^{\frac{(2n+\nu)^2}{4}} e^{\frac{(2n+\nu)i\pi y}{2\Lambda}} e^{\frac{(2m+\mu)i\pi}{2\mathrm{K}}\left(x+2\frac{n+\nu}{2}\log\rho'\right)}$$

$$= \sum_{n=-\infty}^{n=\infty} (-1)^{n\rho} q^{\frac{(2n+\nu)^2}{4}} e^{\frac{(n+\nu)i\pi y}{2\Lambda}} \theta_{\mu,\lambda}\left(x + \frac{2n+\nu}{2}\log\rho'\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91),$$

which is, in effect, ROSENHAIN's definition of the double theta-functions. Taking a particular case,

$$\vartheta_0 = \sum_{n=-\infty}^{n=\infty} q^{n^2} e^{2n\frac{i\pi y}{2\Lambda}} \theta_{0,0}(x + n \log \rho')$$

$$= \theta_{0,0}(x) + q \cos 2y \frac{\pi}{2\Lambda}\{\theta_{0,0}(x + \log\rho') + \theta_{0,0}(x - \log\rho')\}$$

$$+ q^4 \cos 4y \frac{\pi}{2\Lambda}\{\theta_{0,0}(x + 2\log\rho') + \theta_{0,0}(x - 2\log\rho')\}$$

$$+ q^9 \cos 6y \frac{\pi}{2\Lambda}\{\theta_{0,0}(x + 3\log\rho') + \theta_{0,0}(x - 3\log\rho')\}$$

$$+ \ldots$$

$$+ iq \sin 2y \frac{\pi}{2\Lambda}\{\theta_{0,0}(x + \log\rho') - \theta_{0,0}(x - \log\rho')\}$$

$$+ iq^4 \sin 4y \frac{\pi}{2\Lambda}\{\theta_{0,0}(x + 2\log\rho') - \theta_{0,0}(x - 2\log\rho')\}$$

$$+ iq^9 \sin 6y \frac{\pi}{2\Lambda}\{\theta_{0,0}(x + 3\log\rho') - \theta_{0,0}(x - 3\log\rho')\}$$

$$+ \ldots$$

Expanding by TAYLOR's theorem and re-arranging, this gives

$$\vartheta_0 = \theta_{0,0}(x)\left[1 + 2q \cos 2y \frac{\pi}{2\Lambda} + 2q^4 \cos 4y \frac{\pi}{2\Lambda} + 2q^9 \cos 6y \frac{\pi}{2\Lambda} + \ldots\right]$$

$$+ i \log \rho' \frac{d\theta_{0,0}(x)}{dx}\left[2q \sin 2y \frac{\pi}{2\Lambda} + 2.2q^4 \sin 4y \frac{\pi}{2\Lambda} + 3.2q^9 \sin 6y \frac{\pi}{2\Lambda} + \ldots\right]$$

$$+ (\log\rho')^2 \frac{1}{2!} \frac{d^2\theta_{0,0}(x)}{dx^2}\left[2q \cos 2y \frac{\pi}{2\Lambda} + 2^2.2q^4 \cos 4y \frac{\pi}{2\Lambda} + 3^2.2q^9 \cos 6y \frac{\pi}{2\Lambda} + \ldots\right]$$

$$+ i(\log\rho')^3 \frac{1}{3!} \frac{d^3\theta_{0,0}(x)}{dx^3}\left[2q \sin 2y \frac{\pi}{2\Lambda} + 2^3.2q^4 \sin 4y \frac{\pi}{2\Lambda} + 3^3.2q^9 \sin 6y \frac{\pi}{2\Lambda} + \ldots\right]$$

$$+ \ldots$$

The bracket in the first term $= \theta_{0,0}(y)$

" " second " $= -\frac{\Lambda}{\pi}\frac{d\theta_{0,0}(y)}{dy}$

" " third " $= -\left(\frac{\Lambda}{\pi}\right)^2\frac{d^2\theta_{0,0}(y)}{dy^2}$

" " fourth " $= \left(\frac{\Lambda}{\pi}\right)^3\frac{d^3\theta_{0,0}(y)}{dy^3}$

" " n^{th} " $= \pm\left(\frac{\Lambda}{\pi}\right)^n\frac{d^n\theta_{0,0}(y)}{dy^n}$,

the sign being + if $n=4p$ or $4p+1$, and − if $n=4p+2$ or $4p+3$.

Since

$$\log r = \frac{i\pi}{2\mathrm{K}}\log\rho'$$

$$\therefore \log\rho' = \frac{2\mathrm{K}}{\pi}\cdot\frac{1}{i}\cdot\log r$$

Hence

$$\vartheta_0 = \theta_{0,0}(x)\theta_{0,0}(y) - \frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d\theta_{0,0}(x)}{dx}\frac{d\theta_{0,0}(y)}{dy} + \left(\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\right)^2\frac{1}{2!}\frac{d^2\theta_{0,0}(x)}{dx^2}\frac{d^2\theta_{0,0}(y)}{dy^2} + \ldots$$

$$+(-1)^s\left(\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\right)^s\frac{1}{s!}\frac{d^s\theta_{0,0}(x)}{dx^s}\frac{d^s\theta_{0,0}(y)}{dy^s} + \ldots \quad . \quad . \quad (92)$$

which may be expressed in the symbolical form

$$\vartheta_0 = e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d^2}{dxdy}}\theta_{0,0}(x)\theta_{0,0}(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93);$$

and it may be proved by an exactly similar process to hold for all functions, so that generally

$$\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\} = e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d^2}{dxdy}}\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

the parameters of $\theta_{\mu,\lambda}(x)$ being p, K; and of $\theta_{\nu,\rho}(y)$ q, Λ.

25. The functions Φ have already been distinguished by the oddness or evenness of $\mu\lambda+\nu\rho$; this formula (94) enables us to verify the division of the even and uneven functions given in the table. The latter will be obtained by taking one of the single θ's even and the other uneven; and since there are three even and one uneven function θ for each of the variables, there will be six uneven functions Φ, obtained by taking the uneven function of each variable with the three even functions of the other variable: hence there will be ten even functions, since there are 16 $(=4^2)$ in all.

26. The periodicity also easily follows. Since $\theta(x)$ and all its differentials are periodical in 4K and $\theta(y)$ and all its differentials in 4Λ, we have

$$\begin{aligned}\Phi\{x+4m'\mathrm{K},\, y+4n'\Lambda\} &= e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d^2}{dxdy}}\theta_{\mu,\lambda}(x+4m'\mathrm{K})\theta_{\nu,\rho}(y+4n'\Lambda)\\ &= e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d^2}{dxdy}}\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}(y)\\ &= \Phi\{x,y\} \quad \ldots \quad (95)\end{aligned}$$

giving the two pairs of conjugate periods 4K and 0, 0 and 4Λ.

27. If K′ be defined by the relation

$$-\pi\frac{\mathrm{K}'}{\mathrm{K}} = \log p$$

then it is known that

$$\theta_{\mu,\lambda}(x+4i\mathrm{K}') = p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\theta_{\mu,\lambda}(x)$$

Hence

$$\Phi\{x+4i\mathrm{K}',\, y\} = e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d^2}{dxdy}}p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}(y)$$

$$\begin{aligned}=p^{-4}\Big[&e^{-\frac{2\pi ix}{\mathrm{K}}}\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}(y) - \frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d\theta_{\nu,\rho}(y)}{dy}\frac{d}{dx}\Big\{e^{-\frac{2\pi ix}{\mathrm{K}}}\theta_{\mu,\lambda}(x)\Big\}\\ &+\left(\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\right)^2\frac{1}{2!}\frac{d^2\theta_{\nu,\rho}(y)}{dy^2}\frac{d^2}{dx^2}\Big\{e^{-\frac{2\pi ix}{\mathrm{K}}}\theta_{\mu,\lambda}(x)\Big\} - \ldots\Big]\end{aligned}$$

$$\begin{aligned}&=p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\theta_{\mu,\lambda}(x)\left[\theta_{\nu,\rho}(y) - \frac{4\Lambda}{\pi i}\log r\frac{d\theta_{\nu,\rho}(y)}{dy} + \left(\frac{4\Lambda}{\pi i}\log r\right)^2\frac{1}{2!}\frac{d^2\theta_{\nu,\rho}(y)}{dy^2} - \ldots\right]\\ &-p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\frac{d\theta_{\mu,\lambda}(x)}{dx}\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\left[\frac{d\theta_{\nu,\rho}(y)}{dy} - \frac{4\Lambda}{\pi i}\log r\frac{d^2\theta_{\nu,\rho}(y)}{dy^2} + \left(\frac{4\Lambda}{\pi i}\log r\right)^2\frac{1}{2!}\frac{d^3\theta_{\nu,\rho}(y)}{dy^3} - \ldots\right]\\ &+p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\frac{d^2\theta_{\mu,\lambda}(x)}{dx^2}\frac{1}{2!}\left(\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\right)^2\left[\frac{d^2\theta_{\nu,\rho}(y)}{dy^2} - \frac{4\Lambda}{\pi i}\log r\frac{d^3\theta_{\nu,\rho}(y)}{dy^3} + \left(\frac{4\Lambda}{\pi i}\log r\right)^2\frac{1}{2!}\frac{d^4\theta_{\nu,\rho}(y)}{dy^2} - \ldots\right]\\ &-\ldots\end{aligned}$$

$$=p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\left[\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}\left(y-\frac{4\Lambda}{\pi i}\log r\right) - \frac{2\mathrm{K}\Lambda}{\pi^2}\log r\frac{d\theta_{\mu,\lambda}(x)}{dx}\frac{d\theta_{\nu,\rho}\left(y-\frac{4\Lambda}{\pi i}\log r\right)}{dy} + \ldots\right]$$

and therefore

$$\begin{aligned}\Phi\left\{x+4i\mathrm{K}',\, y+\frac{4\Lambda}{\pi i}\log r\right\} = p^{-4}e^{-\frac{2\pi ix}{\mathrm{K}}}\Big[&\theta_{\mu,\lambda}(x)\theta_{\nu,\rho}(y) - \frac{2\mathrm{K}\Lambda\log r}{\pi^2}\frac{d\theta_{\mu,\lambda}(x)}{dx}\frac{d\theta_{\nu,\rho}(y)}{dy}\\ &+\left(\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\right)^2\frac{1}{2!}\frac{d^2\theta_{\mu,\lambda}(x)}{dx^2}\frac{d^2\theta_{\nu,\rho}(y)}{dy^2} - \ldots\Big]\end{aligned}$$

that is

$$\Phi\left\{x+\frac{4\mathrm{K}}{\pi i}\log p,\ y+\frac{4\Lambda}{\pi i}\log r\right\}=p^{-1}e^{-\frac{2\pi ix}{\mathrm{K}}}\Phi\{x,\,y\}.$$

Similarly

$$\Phi\left\{x+\frac{4\mathrm{K}}{\pi i}\log r,\ y+\frac{4\Lambda}{\pi i}\log q\right\}=q^{-1}e^{-\frac{2\pi iy}{\Lambda}}\Phi\{x,\,y\} \quad \ldots \quad (96).$$

which give the two pairs of conjugate quasi-periods,

$$\frac{4\mathrm{K}}{\pi i}\log p \text{ and } \frac{4\Lambda}{\pi i}\log r,\quad \frac{4\mathrm{K}}{\pi i}\log r \text{ and } \frac{4\Lambda}{\pi i}\log q.$$

28. The verification of the expansions in doubly infinite series of sines and cosines is easily effected: for substituting in (92) the expressions

$$\theta_{0,0}(x)=1+2p\cos\frac{\pi x}{\mathrm{K}}+2p^4\cos\frac{2\pi x}{\mathrm{K}}+2p^9\cos\frac{3\pi x}{\mathrm{K}}+\ldots$$

$$\theta_{0,0}(y)=1+2q\cos\frac{\pi y}{\Lambda}+2q^4\cos\frac{2\pi y}{\Lambda}+2q^9\cos\frac{3\pi y}{\Lambda}+\ldots$$

the coefficient, on the right hand side, of $\cos\pi\left(\frac{mx}{\mathrm{K}}-\frac{ny}{\Lambda}\right)$ is

$$=2\left[p^{m^2}q^{n^2}-2mn\log r.p^{m^2}q^{n^2}+\frac{(2\log r)^2}{2!}m^2n^2p^{m^2}q^{n^2}-\ldots\right]$$

$$=2p^{m^2}q^{n^2}r^{-2mn}$$

which is right.

Second proof of the product theorem (23).

29. The product theorem for single theta-functions, as given by Professor SMITH (Lond. Math. Soc. Proc., vol. i.) is, with the notation of Section I,

$$2\Pi\theta_{\mu,\lambda}(x)=\Pi\theta_{\sigma,\mathrm{A}}(\mathrm{X})+\Pi\theta_{\sigma,\mathrm{A}+1}(\mathrm{X})+(-1)^{\mathrm{A}'}\{\Pi\theta_{\sigma+1,\mathrm{A}}(\mathrm{X})-\Pi\theta_{\sigma+1,\mathrm{A}+1}(\mathrm{X})\} \quad . \quad (97)$$

Now, using (94), we have

$$\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x,\,y\right\}\Phi\left\{\begin{pmatrix}\lambda', \rho'\\ \mu', \nu'\end{pmatrix}x',\,y'\right\}=e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\left(\frac{d^2}{dxdy}+\frac{d^2}{dx'dy'}\right)}\theta_{\mu,\lambda}(x)\theta_{\mu',\lambda'}(x')\theta_{\nu,\rho}(y)\theta_{\nu',\rho}(y')$$

and therefore

$$\Pi\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x,\,y\right\}=e^{-\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\left(\frac{d^2}{dx_1dy_1}+\frac{d^2}{dx_2dy_2}+\frac{d^2}{dx_3dy_3}+\frac{d^2}{dx_4dy_4}\right)}\Pi\theta_{\mu,\lambda}(x)\Pi\theta_{\nu,\rho}(y) \quad . \quad . \quad (98)$$

By the values of X, Y we have

$$\frac{d}{dx_1}=\tfrac{1}{2}\left(-\frac{d}{dX_1}+\frac{d}{dX_2}+\frac{d}{dX_3}+\frac{d}{dX_4}\right)$$

$$\frac{d}{dx_2}=\tfrac{1}{2}\left(\frac{d}{dX_1}-\frac{d}{dX_2}+\frac{d}{dX_3}+\frac{d}{dX_4}\right)$$

$$\cdots\cdots$$

$$\frac{d}{dy_1}=\tfrac{1}{2}\left(-\frac{d}{dY_1}+\frac{d}{dY_2}+\frac{d}{dY_3}+\frac{d}{dY_4}\right)$$

$$\cdots\cdots$$

hence

$$\frac{d^2}{dx_1dy_1}+\frac{d^2}{dx_2dy_2}+\frac{d^2}{dx_3dy_3}+\frac{d^2}{dx_4dy_4}=\frac{d^2}{dX_1dY_1}+\frac{d^2}{dX_2dY_2}+\frac{d^2}{dX_3dY_3}+\frac{d^2}{dX_4dY_4}\quad . \quad . \quad (99).$$

By means of (97) and the corresponding theorem for $\Pi\theta_{\nu,\rho}(y)$, an expression is obtained for $\Pi\theta_{\mu,\lambda}(x)\Pi\theta_{\nu,\rho}(y)$, containing 16 terms; substitute this in (98) and transpose the operator by (99), and then, by (94), express each term as the product of four Φ's, and there will result the theorem already given in (23).

On the differential equations satisfied by $\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}$.

30. From the theory of elliptic functions it is known that if

$$K=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}}$$

$$\Lambda=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\lambda^2\sin^2\theta}}$$

K, Λ, k, λ are definite functions of p, q; viz.:

$$\sqrt{\frac{2K}{\pi}}=1+2p+2p^4+2p^9+\ldots$$

$$\sqrt{\frac{2\Lambda}{\pi}}=1+2q+2q^4+2q^9+\ldots$$

$$\sqrt{\kappa}=\frac{2p^{\frac{1}{4}}+2p^{\frac{9}{4}}+2p^{\frac{25}{4}}+\ldots}{1+2p+2p^4+2p^9+\ldots}$$

$$\sqrt{\lambda}=\frac{2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+\ldots}{1+2q+2q^4+2q^9+\ldots}$$

Also K, κ are given each as a function of the other by the respective differential equations

$$(1-\kappa^2)\frac{d^2\mathrm{K}}{d\kappa^2}+\frac{1-3\kappa^2}{\kappa}\frac{d\mathrm{K}}{d\kappa}-\mathrm{K}=0$$

$$(1-\kappa^2)\frac{d^2\kappa}{d\mathrm{K}^2}-\frac{1-3\kappa^2}{\kappa}\left(\frac{d\kappa}{d\mathrm{K}}\right)^2+\mathrm{K}\left(\frac{d\kappa}{d\mathrm{K}}\right)^3=0,$$

so that κ may be considered known, and likewise κ', E, given by

$$\kappa^2+\kappa'^2=1, \qquad \mathrm{E}=\int_0^{\frac{\pi}{2}}\sqrt{1-\kappa^2\sin^2\theta}\,d\theta.$$

Similarly if

$$\lambda'^2+\lambda^2=1,$$

$$\Gamma=\int_0^{\frac{\pi}{2}}\sqrt{1-\lambda^2\sin^2\theta}\,d\theta,$$

λ', Γ may be considered known.

31. It is proved in CAYLEY's 'Elliptic Functions,' § 310, that the general single theta-function satisfies the differential equation

$$\frac{d^2\theta}{dx^2}-2x\left(\kappa'^2-\frac{\mathrm{E}}{\mathrm{K}}\right)\frac{d\theta}{dx}+2\kappa\kappa'^2\frac{d\theta}{d\kappa}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

and

$$\frac{d\mathrm{K}}{dk}=\frac{\mathrm{E}}{\kappa\kappa'^2}-\frac{\mathrm{K}}{\kappa}=-\frac{\mathrm{K}}{\kappa\kappa'^2}\left(\kappa'^2-\frac{\mathrm{E}}{\mathrm{K}}\right)$$

so that (100) may be written in the form

$$\frac{d^2\theta}{dx^2}+\frac{2\kappa\kappa'^2}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}x\frac{d\theta}{dx}+2\kappa\kappa'^2\frac{d\theta}{d\kappa}=0.$$

Differentiating s times with respect to x

$$\frac{d^2}{dx^2}\frac{d^s\theta}{dx^s}+\frac{2\kappa\kappa'^2}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}x\frac{d}{dx}\frac{d^s\theta}{dx^s}+2s\frac{\kappa\kappa'^2}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}\frac{d^s\theta}{dx^s}+2\kappa\kappa'^2\frac{d}{d\kappa}\frac{d^s\theta}{dx^s}=0 \quad . \quad . \quad . \quad (101)$$

Now the general term in Φ in (94) is a numerical multiple of

$$\psi'=\left(\frac{2\mathrm{K}\Lambda\log r}{\pi^2}\right)^s\frac{d^s\theta(x)}{dx^s}\frac{d^s\theta(y)}{dy^s}$$

or of

$$\psi=\mathrm{K}^s\frac{d^s\theta}{dx^s}$$

so far as x, κ are concerned. Then

$$\frac{d^2\psi}{dx^2}+\frac{2\kappa\kappa'^2}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}x\frac{d\psi}{dx}=\mathrm{K}^s\,[\text{first two terms in (101)}]$$

and

$$\frac{d\psi}{d\kappa}=\mathrm{K}^s\left[\frac{d}{d\kappa}\frac{d^s\theta}{dx^s}+\frac{s}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}\frac{d^s\theta}{dx^s}\right]$$

Hence

$$\frac{d^2\psi}{dx^2}+\frac{2\kappa\kappa'^2}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}x\frac{d\psi}{dx}+2\kappa\kappa'^2\frac{d\psi}{d\kappa}=0$$

and therefore also Φ, the sum of the terms ψ, satisfies the differential equation

$$\frac{d^2\Phi}{dx^2}+\frac{2\kappa\kappa'^2}{\mathrm{K}}\frac{d\mathrm{K}}{d\kappa}x\frac{d\Phi}{dx}+2\kappa\kappa'^2\frac{d\Phi}{d\kappa}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

or restoring the usual coefficient in the second term

$$\frac{d^2\Phi}{dx^2}-2x\left(\kappa'^2-\frac{\mathrm{E}}{\mathrm{K}}\right)\frac{d\Phi}{dx}+2\kappa\kappa'^2\frac{d\Phi}{d\kappa}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

Similarly Φ satisfies

$$\frac{d^2\Phi}{dy^2}-2y\left(\lambda'^2-\frac{\Gamma}{\Lambda}\right)\frac{d\Phi}{dy}+2\lambda\lambda'^2\frac{d\Phi}{d\lambda}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104),$$

or

$$\frac{d^2\Phi}{dy^2}+\frac{2\lambda\lambda'^2}{\Lambda}\frac{d\Lambda}{d\lambda}y\frac{d\Phi}{dy}+2\lambda\lambda'^2\frac{d\Phi}{d\lambda}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105);$$

and from (94) it at once follows that

$$r\frac{d\Phi}{dr}+\frac{2\mathrm{K}\Lambda}{\pi^2}\frac{d^2\Phi}{dxdy}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106).$$

32. All these equations can be deduced from the general definition of Φ, viz.:—

$$\Phi=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{(2m+\mu)^2}{4}}q^{\frac{(2n+\nu)^2}{4}}r^{\frac{(2m+\mu)(2n+\nu)}{2}}e^{\frac{i\pi}{2}\left\{(2m+\mu)\frac{x}{\mathrm{K}}+(2n+\nu)\frac{y}{\Lambda}\right\}}$$

The equation (106) is obviously satisfied. Consider the general term in Φ to obtain (102); it is a multiple of

$$u=p^{\frac{(2m+\mu)^2}{4}}e^{\frac{i\pi x}{2\mathrm{K}}(2m+\mu)}$$

the coefficient being independent of x, κ. Now

$$\frac{d^2u}{dx^2}=-\frac{(2m+\mu)^2\pi^2}{4K^2}u$$

$$\frac{du}{dp}=\frac{(2m+\mu)^2}{4p}u-\frac{i\pi x(2m+\mu)}{2K^2}\frac{dK}{dp}u$$

and

$$x\frac{du}{dx}=\frac{i\pi x(2m+\mu)}{2K}u$$

Hence

$$\frac{du}{dp}=-\frac{K^2}{\pi^2 p}\frac{d^2u}{dx^2}-\frac{1}{K}\frac{dK}{dp}x\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Also

$$p=e^{-\pi\frac{K'}{K}}$$

$$\therefore \frac{1}{p}\frac{dp}{d\kappa}=-\pi\frac{K\frac{dK'}{d\kappa}-K'\frac{dK}{d\kappa}}{K^2}$$

$$=-\frac{\pi}{\kappa\kappa'^2K^2}\{-KE'-K'E+KK'\}$$

$$=\frac{\pi^2}{2\kappa\kappa'^2K^2}$$

Multiplying (107) throughout by $\frac{dp}{d\kappa}$ and substituting in the first term on the right-hand side the value just found for $\frac{1}{p}\frac{dp}{d\kappa}$, we have

$$\frac{du}{dk}=-\frac{1}{2\kappa\kappa'^2}\frac{d^2u}{dx^2}-\frac{1}{K}\frac{dK}{d\kappa}x\frac{du}{dx}$$

and hence Φ, the sum of the terms u, satisfies

$$\frac{d^2\Phi}{dx^2}+\frac{2\kappa\kappa'^2}{K}\frac{dK}{d\kappa}x\frac{d\Phi}{dx}+2\kappa\kappa'^2\frac{d\Phi}{d\kappa}=0$$

which is (102). The quantity $\frac{dK}{d\kappa}$ may be explicitly expressed in the terms of p as follows. We have

$$\theta_{0,0}\left(x\frac{2K}{\pi}\right)=1-p^2.1-p^4.1-p^6 \ldots (1+2p\cos 2x+p^2)(1+2p^3\cos 2x+p^6)\ldots$$

and

$$\theta_{0,0}(0)=\left(\frac{2K}{\pi}\right)^{\frac{1}{2}}$$

Writing u for $\frac{2\mathrm{K}x}{\pi}$ and taking logarithmic differentials

$$-\tfrac{1}{4}\frac{1}{\theta_{0,0}(u)}\frac{2\mathrm{K}}{\pi}\frac{d\theta_{0,0}(u)}{du}=\frac{p\sin 2x}{1+2p\cos 2x+p^2}+\frac{p^3\cos 2x}{1+2p^3\cos 2x+p^6}+\ldots$$

$$=\sin 2x\left[\frac{p}{1+2p\cos 2x+p^2}+\frac{p^3}{1+2p^3\cos 2x+p^6}+\frac{p^5}{1+2p^5\cos 2x+p^{10}}+\ldots\right]$$

Differentiate again with regard to x, and then put x zero:

$$-\tfrac{1}{4}\left(\frac{\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}\left(\frac{2\mathrm{K}}{\pi}\right)^2\frac{d^2\theta_{0,0}(u)}{du_0^{\,2}}=2\left[\frac{p}{(1+p)^2}+\frac{p^3}{(1+p^3)^2}+\frac{p^5}{(1+p^5)^2}+\ldots\right]$$

But by (100)

$$-\frac{d^2\theta_{0,0}(x)}{dx_0^{\,2}}=2\kappa\kappa'^2\frac{d\theta_{0,0}(0)}{d\kappa}$$

$$=2\kappa\kappa'^2\sqrt{\frac{1}{2\pi\mathrm{K}}}\frac{d\mathrm{K}}{d\kappa}$$

Hence

$$\frac{\kappa\kappa'^2\mathrm{K}}{2\pi^2}\frac{d\mathrm{K}}{d\kappa}=\frac{p}{(1+p)^2}+\frac{p^3}{(1+p^3)^2}+\frac{p^5}{(1+p^5)^2}+\frac{p^7}{(1+p^7)^2}+\ldots$$

so that now all the coefficients in (102) are known explicitly in terms of p.

On the constants.

33. From the definitions of ϑ_0, c_0 we have

$$\begin{aligned}c_0=&1+2p+2p^4+2p^2+\ldots+2q+2q^4+2q^9+\ldots\\&+2p\left\{q\left(r^2+\frac{1}{r^2}\right)+q^4\left(r^4+\frac{1}{r^4}\right)+q^9\left(r^6+\frac{1}{r^6}\right)+\ldots\right\}\\&+2p^4\left\{q\left(r^4+\frac{1}{r^4}\right)+q^4\left(r^8+\frac{1}{r^8}\right)+q^9\left(r^{12}+\frac{1}{r^{12}}\right)+\ldots\right\}\\&+2p^9\left\{q\left(r^6+\frac{1}{r^6}\right)+q^4\left(r^{12}+\frac{1}{r^{12}}\right)+q^9\left(r^{18}+\frac{1}{r^{12}}\right)+\ldots\right\}\\&+\ldots\end{aligned}$$

with similar series for c_1, c_2, c_3, c_4, c_6, c_8, c_9, c_{12}, c_{15}. Substituting these in the complete set of sixty equations of which (41) are the type, there will result algebraical identities in three quantities p, q, r (mutually independent), corresponding to identities in the "q-series" in elliptic functions. As an example

$$\Big[1+2q+2q^4+2q^9+\ldots+2p\left\{1+q\left(r^2+\frac{1}{r^2}\right)+q^4\left(r^4+\frac{1}{r^4}\right)+q^9\left(r^6+\frac{1}{r^6}\right)+\ldots\right\}$$
$$+2p^4\left\{1+q\left(r^4+\frac{1}{r^4}\right)+q^4\left(r^8+\frac{1}{r^8}\right)+q^9\left(r^{12}+\frac{1}{r^{12}}\right)+\ldots\right\}$$
$$+2p^9\left\{1+q\left(r^6+\frac{1}{r^6}\right)+q^4\left(r^{12}+\frac{1}{r^{12}}\right)+\ldots\right\}+\ldots\Big]^4$$

$$=\Big[2p^{\frac{1}{4}}\left\{1+q\left(r+\frac{1}{r}\right)+q^4\left(r^2+\frac{1}{r^2}\right)+q^9\left(r^3+\frac{1}{r^3}\right)+\ldots\right\}$$
$$+2p^{\frac{9}{4}}\left\{1+q\left(r^3+\frac{1}{r^3}\right)+q^4\left(r^6+\frac{1}{r^6}\right)+q^9\left(r^9+\frac{1}{r^9}\right)+\ldots\right\}$$
$$+2p^{\frac{25}{4}}\left\{1+q\left(r^5+\frac{1}{r^5}\right)+q^4\left(r^{10}+\frac{1}{r^{10}}\right)+q^9\left(r^{15}+\frac{1}{r^{15}}\right)+\ldots\right\}+\ldots\Big]^4$$

$$+\Big[2q^{\frac{1}{4}}\left\{1-p\left(r+\frac{1}{r}\right)+p^4\left(r^2+\frac{1}{r^2}\right)-p^9\left(r^3+\frac{1}{r^3}\right)+\ldots\right\}$$
$$+2q^{\frac{9}{4}}\left\{1-p\left(r^3+\frac{1}{r^3}\right)+p^4\left(r^6+\frac{1}{r^6}\right)-p^9\left(r^9+\frac{1}{r^9}\right)+\ldots\right\}$$
$$+2q^{\frac{25}{4}}\left\{1-p\left(r^5+\frac{1}{r^5}\right)+p^4\left(r^{10}+\frac{1}{r^{10}}\right)-p^9\left(r^{15}+\frac{1}{r^{15}}\right)+\ldots\right\}+\ldots\Big]^4$$

$$+\Big[1-2q+2q^4-2q^9+\ldots-2p\left\{1-q\left(r^2+\frac{1}{r^2}\right)+q^4\left(r^4+\frac{1}{r^4}\right)-q^9\left(r^6+\frac{1}{r^6}\right)+\ldots\right\}$$
$$+2p^4\left\{1-q\left(r^4+\frac{1}{r^4}\right)+q^4\left(r^8+\frac{1}{r^8}\right)-q^9\left(r^{12}+\frac{1}{r^{12}}\right)+\ldots\right\}$$
$$-2p^9\left\{1-q\left(r^6+\frac{1}{r^6}\right)+q^4\left(r^{12}+\frac{1}{r^{12}}\right)-q^9\left(r^{18}+\frac{1}{r^{18}}\right)+\ldots\right\}+\ldots\Big]^4$$

Also equations (42), (43), (44) give expressions for κ_1, κ_2, κ_3, in terms of p, q, r; and other identities are obtained from the equation (α)–$(\iota\epsilon)$.

34. Let $\kappa^2=c$, $\kappa'^2=c'$; then θ being any single theta-function

$$\frac{d^2\theta}{dx^2}-2\alpha x\frac{d\theta}{dx}+\beta\frac{d\theta}{dc}=0$$

where

$$\alpha=c'-\frac{\mathrm{E}}{\mathrm{K}}=-\frac{2cc'}{\mathrm{K}}\frac{d\mathrm{K}}{dc},$$

$$\beta=4cc'.$$

First, let θ be one of the three even functions: differentiating $2n$ times with respect to x, and then putting x zero

$$\frac{d^{2n+2}\theta}{dx_0^{\,2n+2}}-4n\alpha\frac{d^{2n}\theta}{dx_0^{\,2n}}+\beta\frac{d}{dc}\frac{d^{2n}\theta}{dx_0^{\,2n}}=0$$

and therefore

$$\frac{1}{4cc'}\frac{d^{2n+2}\theta}{dx_0^{2n+2}}=-\frac{2n}{K}\frac{dK}{dc}\frac{d^{2n}\theta}{dx_0^{2n}}-\frac{d}{dc}\frac{d^{2n}\theta}{dx_0^{2n}}$$

$$=-K^{-2n}\frac{d}{dc}\cdot K^{2n}\frac{d^{2n}\theta}{dx_0^{2n}}$$

that is

$$K^{2n+2}\frac{d^{2n+2}\theta}{dx_0^{2n+2}}=\left(-4cc'K^2\frac{d}{dc}\right)K^{2n}\frac{d^{2n}\theta}{dx_0^{2n}}$$

$$=\left(-4cc'K^2\frac{d}{dc}\right)^{n+1}\theta(0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108).$$

Now

$$\theta_{0,0}(0)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}K^{\frac{1}{2}}$$

$$\theta_{0,1}(0)=\left(\frac{}{\pi}\right)^{\frac{1}{2}}K^{\frac{1}{2}}c'^{\frac{1}{4}}$$

$$\theta_{1,0}(0)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}K^{\frac{1}{2}}c^{\frac{1}{4}}$$

and thus, by (108),

$$K^{2n}\frac{d^{2n}\theta_{0,0}}{dx_0^{2n}}=(-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4cc'K^2\frac{d}{dc}\right)^nK^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109).$$

$$K^{2n}\frac{d^{2n}\theta_{0,1}}{dx_0^{2n}}=(-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4cc'K^2\frac{d}{dc}\right)^nK^{\frac{1}{2}}c'^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110).$$

$$K^{2n}\frac{d^{2n}\theta_{1,0}}{dx_0^{2n}}=(-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4cc'K^2\frac{d}{dc}\right)^nK^{\frac{1}{2}}c^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111).$$

Similarly, if $\lambda^2=\gamma$, $\lambda'^2=\gamma'$,

$$\Lambda^{2n}\frac{d^{2n}\theta_{0,0}}{dy_0^{2n}}=(-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4\gamma\gamma'\Lambda^2\frac{d}{d\gamma}\right)^n\Lambda^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

$$\Lambda^{2n}\frac{d^{2n}\theta_{0,1}}{dy_0^{2n}}=(-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4\gamma\gamma'\Lambda^2\frac{d}{d\gamma}\right)^n\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

$$\Lambda^{2n}\frac{d^{2n}\theta_{1,0}}{dy_0^{2n}}=(-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4\gamma\gamma'\Lambda^2\frac{d}{d\gamma}\right)^n\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114).$$

Next, let θ be the uneven function; differentiating the equation $(2n+1)$ times with regard to x and putting x zero

$$\frac{d^{2n+3}\theta_{1,1}}{dx_0^{2n+3}}-2\alpha(2n+1)\frac{d^{2n+1}\theta_{1,1}}{dx_0^{2n+1}}+\beta\frac{d}{dc}\frac{d^{2n+1}\theta_{1,1}}{dx_0^{2n+1}}=0$$

and therefore

$$\frac{1}{4cc'}\frac{d^{2n+3}\theta_{1,1}}{dx_0^{\,2n+3}}=-\frac{2n+1}{\mathrm{K}}\frac{d\mathrm{K}}{dc}\frac{d^{2n+1}\theta_{1,1}}{dx_0^{\,2n+1}}-\frac{d}{dc}\frac{d^{2n+1}\theta_{1,1}}{dx_0^{\,2n+1}}$$

$$\mathrm{K}^{2n+3}\frac{d^{2n+3}\theta_{1,1}}{dx_0^{\,2n+3}}=-4cc'\mathrm{K}^2\frac{d}{dc}\mathrm{K}^{2n+1}\frac{d^{2n+1}\theta_{1,1}}{dx_0^{\,2n+1}}$$

that is

$$\mathrm{K}^{2n+1}\frac{d^{2n+1}\theta_{1,1}}{dx_0^{\,2n+1}}=\left(-4cc'\mathrm{K}^2\frac{d}{dc}\right)^n\mathrm{K}\frac{d\theta_{1,1}}{dx_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

Now

$$\frac{1}{i}\theta_{1,1}\left(\frac{2\mathrm{K}x}{\pi}\right)=2p^{\frac{1}{4}}\sin x-2p^{\frac{9}{4}}\sin 3x+2p^{\frac{25}{4}}\sin 5x-\ldots$$

$$=2p^{\frac{1}{4}}(1-p^2)(1-p^2)(1-p^6)\ldots\sin x(1-2p^2\cos 2x+p^2)(1-2p^4\cos 2x+p^8)\ldots$$

Hence

$$\frac{\mathrm{K}}{\pi i}\frac{d\theta_{1,1}}{dx_0}=p^{\frac{1}{4}}\{(1-p^2)(1-p^4)(1-p^6\ldots\}^3$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\pi}c^{\frac{1}{4}}c'^{\frac{1}{4}}\mathrm{K}^{\frac{3}{2}}$$

Thus (115) gives

$$\mathrm{K}^{2n+1}\frac{d^{2n+1}\theta_{1,1}}{dx_0^{\,2n+1}}=(-1)^{n+\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4cc'\mathrm{K}^2\frac{d}{dc}\right)^n c^{\frac{1}{4}}c'^{\frac{1}{4}}\mathrm{K}^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (116).$$

Similarly

$$\Lambda^{2n+1}\frac{d^{2n+1}\theta_{1,1}}{dy_0^{\,2n+1}}=(-1)^{n+\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(4\gamma\gamma'\Lambda^2\frac{d}{d\gamma}\right)^n \gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (117).$$

35. Another form may be given to several of these formulæ. Let

$$\left.\begin{aligned}\log p&=p'\\ \log q&=q'\\ 2\log r&=r'\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118).$$

Then

$$\frac{dp'}{d\kappa}=\frac{1}{p}\frac{dp}{d\kappa}=\frac{\pi^2}{2\kappa\kappa'^2\mathrm{K}^2}$$

and therefore

$$4cc'\mathrm{K}^2\frac{d}{dc}=\pi^2\frac{d}{dp'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119),$$

and

$$4\gamma\gamma'\Lambda^2\frac{d}{d\gamma}=\pi^2\frac{d}{dq'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

These formulæ practically contain the expansions in powers of x of the single theta-functions; restoring κ, κ', these are

$$\left.\begin{aligned}
\theta_{0,0}\left(\frac{Kx}{\pi}\right)&=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[K^{\frac{1}{2}}-\frac{x^2}{2!}\frac{d.K^{\frac{1}{2}}}{dp'}+\frac{x^4}{4!}\frac{d^2.K^{\frac{1}{2}}}{dp'^2}-\frac{x^6}{6!}\frac{d^3.K^{\frac{1}{2}}}{dp'^3}+\ldots\right]\\
\theta_{0,1}\left(\frac{Kx}{\pi}\right)&=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[(\kappa'K)^{\frac{1}{2}}-\frac{x^2}{2!}\frac{d.(\kappa'K)^{\frac{1}{2}}}{dp'}+\frac{x^4}{4!}\frac{d^2.(\kappa'K)^{\frac{1}{2}}}{dp'^2}-\frac{x^6}{6!}\frac{d^3.(\kappa'K)^{\frac{1}{2}}}{dp'^3}+\ldots\right]\\
\theta_{1,0}\left(\frac{Kx}{\pi}\right)&=\left(\frac{2}{\pi}\right)\left[(\kappa K)^{\frac{1}{2}}-\frac{x^2}{2!}\frac{d.(\kappa K)^{\frac{1}{2}}}{dp'}+\frac{x^4}{4!}\frac{d^2.(\kappa K)^{\frac{1}{2}}}{dp'^2}-\frac{x^6}{6!}\frac{d^3.(\kappa K)^{\frac{1}{2}}}{dp'^3}+\ldots\right]\\
\frac{1}{i}\,\theta_{1,1}\left(\frac{Kx}{\pi}\right)&=\left(\frac{2}{\pi^3}\right)^{\frac{1}{2}}\left[(\kappa\kappa'K^3)^{\frac{1}{2}}x-\frac{x^3}{3!}\frac{d.(\kappa\kappa'K^3)^{\frac{1}{2}}}{dp'}+\frac{x^5}{5!}\frac{d^2.(\kappa\kappa'K^3)^{\frac{1}{2}}}{dp'^2}-\ldots\right]
\end{aligned}\right\} \quad (121).$$

Expanding the cosines in the right-hand side of

$$\theta_{0,0}\left(\frac{Kx}{\pi}\right)=1+2p\cos x+2p^4\cos 2x+2p^9\cos 3x+\ldots$$

and equating coefficients of x^n, we obtain

$$\frac{d^n.K^{\frac{1}{2}}}{dp'^n}=(2\pi)^{\frac{1}{2}}\{p+4^np^4+9^np^9+16^np^{16}+\ldots\}$$

which is easily deducible from

$$\left(\frac{2K}{\pi}\right)^{\frac{1}{2}}=1+2p+2p^4+2p^9+\ldots$$

and so verifies the above expansions.

Similarly

$$\left.\begin{aligned}
\frac{d^n.(\kappa'K)^{\frac{1}{2}}}{dp'^n}&=(2\pi)^{\frac{1}{2}}\{-p+4^np^4-9^np^9+16^np^{16}-\ldots\}\\
\frac{d^n.(\kappa K)^{\frac{1}{2}}}{dp'^n}&=(2\pi)^{\frac{1}{2}}\left\{\frac{1}{4^n}p^{\frac{1}{4}}+\left(\frac{9}{4}\right)^n p^{\frac{9}{4}}+\left(\frac{25}{4}\right)^n p^{\frac{25}{4}}+\ldots\right\}\\
\frac{d^n.(\kappa\kappa'K^3)^{\frac{1}{2}}}{dp'^n}&=(2\pi^3)^{\frac{1}{2}}\left\{\frac{1}{4^n}p^{\frac{1}{4}}-\left(\frac{9}{4}\right)^n p^{\frac{9}{4}}+\left(\frac{25}{4}\right)^n p^{\frac{25}{4}}-\ldots\right\}
\end{aligned}\right\} \quad (122).$$

36. By means of the same formulæ it is possible to obtain expressions for all the constant coefficients which arise in the expansions of all the ϑ's in powers of x and y. Since by formula (4)

$$\vartheta_0(x,y)=\vartheta_0(-x,-y)$$

it follows that

$$\vartheta_0=c_0-\frac{1}{2!}(B_{0,0},B_{0,1},B_{0,2}\between x,y)^2+\ldots$$

$$+\frac{(-1)^n}{2n!}(N_{0,0},N_{0,1},N_{0,2},\ldots,N_{0,r},\ldots,N_{0,2n}\between x,y)^{2n}+\ldots \quad (123)$$

where

$$(-1)^nN_{0,s}=\frac{d^{2n}\vartheta_0}{dx_0^{2n-s}dy_0^{\ s}}$$

the zero subscript in the differential coefficient implying that the variables are made to vanish after differentiation. Now

$$\vartheta_0=\theta_{0,0}(x)\theta_{0,0}(y)-\frac{r'\mathrm{K}\Lambda}{\pi^2}\frac{d\theta_{0,0}}{dx}\frac{d\theta_{0,0}}{dy}+\left(\frac{r'\mathrm{K}\Lambda}{\pi^2}\right)^2\frac{1}{2!}\frac{d^2\theta_{0,0}}{dx^2}\frac{d^2\theta_{0,0}}{dy^2}-\ldots$$

Remembering that an uneven differential of an even function is an uneven function and that an even differential is an even function, we have

$$(-1)^n\mathrm{N}_{0,2s}=\frac{d^{2n-2s}\theta_{0,0}}{dx_0^{2n-2s}}\frac{d^{2s}\theta_{0,0}}{dy_0^{2s}}+\frac{1}{2!}\left(\frac{r'\mathrm{K}\Lambda}{\pi^2}\right)^2\frac{d^{2n-2s+2}\theta_{0,0}}{dx_0^{2n-2s+2}}\frac{d^{2s+2}\theta_{0,0}}{dy_0^{2s+2}}+\ldots$$

$$(-1)^n\mathrm{N}_{0,2s+1}=-\frac{r'\mathrm{K}\Lambda}{\pi^2}\frac{d^{2n-2s}\theta_{0,0}}{dx_0^{2n-2s}}\frac{d^{2s+2}\theta_{0,0}}{dy_0^{2s+2}}-\frac{1}{3!}\left(\frac{r'\mathrm{K}\Lambda}{\pi^2}\right)^3\frac{d^{2n-2s+2}\theta_{0,0}}{dx_0^{2n-2s+2}}\frac{d^{2s+4}\theta_{0,0}}{dy_0^{2s+4}}-\ldots$$

Let

$$\Delta_1=\frac{2}{\pi}\left[1+\frac{r'^2}{2!}\frac{d^2}{dp'dq'}+\frac{r'^4}{4!}\frac{d^4}{dp'^2dq'^2}+\ldots\right]$$

$$=\frac{2}{\pi}\cosh\left[r'\left(\frac{d^2}{dp'dq'}\right)^{\frac{1}{2}}\right] \quad \ldots \quad (124),$$

$$\Delta_2=\frac{2}{\pi}\left[1+\frac{r'^2}{3!}\frac{d^2}{dp'dq'}+\frac{r'^4}{5!}\frac{d^4}{dp'^2dq'^2}+\ldots\right]$$

$$=\frac{2}{\pi r'\left(\frac{d^2}{dp'dq'}\right)^{\frac{1}{2}}}\sinh\left[r'\left(\frac{d^2}{dp'dq'}\right)^{\frac{1}{2}}\right] \quad \ldots \quad (125).$$

Then we have

$$c_0=\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}} \quad \ldots \quad (126),$$

$$\mathrm{N}_{0,2s}=\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}.c_0 \quad \ldots \quad (127),$$

and

$$\mathrm{N}_{0,2s+1}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}} \quad \ldots \quad (128).$$

The remaining formulæ are obtained by an exactly similar process and are as follow.

$$\vartheta_1=c_1-\frac{1}{2!}(\mathrm{B}_{1,0},\ \mathrm{B}_{1,1},\ \mathrm{B}_{1,2})(x,\ y)^2+\ldots$$

$$+\frac{(-1)^n}{2n!}(\mathrm{N}_{1,0},\ \mathrm{N}_{1,1},\ldots,\ \mathrm{N}_{1,s},\ldots,\ \mathrm{N}_{1,2n})(x,\ y)^{2n}+\ldots \quad \ldots \quad (129),$$

where

$$c_1=\Delta_1.\mathrm{K}^{\frac{1}{2}}c^{\frac{1}{2}}\Lambda^{\frac{1}{2}} \quad \ldots \quad (130),$$

$$\mathrm{N}_{1,2s}=\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}c_1 \quad \ldots \quad (131),$$

and

$$\mathrm{N}_{1,2s+1}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2.\mathrm{K}^{\frac{1}{2}}c^{\frac{1}{2}}\Lambda^{\frac{1}{2}} \quad \ldots \quad (132).$$

$$\vartheta_2=c_2-\frac{1}{2\,!}(B_{2,0},\ B_{2,1},\ B_{2,2})(x,\ y)^2+\ldots$$

$$+\frac{(-1)^n}{2n\,!}(N_{2,0},\ N_{2,1},\ N_{2,2},\ \ldots,\ N_{2,s},\ \ldots,\ N_{2,2n})(x,\ y)^{2n}+\ldots \quad (133),$$

where

$$c_2=\Delta_1.K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}} \quad (134),$$

$$N_{2,2s}=\left(\frac{\pi}{K}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}c_2 \quad (135),$$

and

$$N_{2,2s+1}=\frac{r'K\Lambda}{\pi^2}\left(\frac{\pi}{K}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2.K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}} \quad (136).$$

$$\vartheta_3=c_3-\frac{1}{2\,!}(B_{3,0},\ B_{3,1},\ B_{3,2})(x,\ y)^2+\ldots$$

$$+\frac{(-1)^n}{2n\,!}(N_{3,0},\ N_{3,1},\ N_{3,2},\ \ldots,\ N_{3,s},\ \ldots,\ N_{3,2n})(x,\ y)^{2n}+\ldots \quad (137),$$

where

$$c_3=\Delta_1.K^{\frac{1}{2}}c^{\frac{1}{4}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}} \quad (138),$$

$$N_{3,2s}=\left(\frac{\pi}{K}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}c_3 \quad (139),$$

$$N_{3,2s+1}=\frac{r'K\Lambda}{\pi^2}\left(\frac{\pi}{K}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2.K^{\frac{1}{2}}c^{\frac{1}{4}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}} \quad (140).$$

$$\vartheta_4=c_4-\frac{1}{2\,!}(B_{4,0},\ B_{4,1},\ B_{4,2})(x,\ y)^2+\ldots$$

$$+\frac{(-1)^n}{2n\,!}(N_{4,0},\ N_{4,1},\ N_{4,2},\ \ldots,\ N_{4,s},\ \ldots,\ N_{4,2n})(x,\ y)^{2n}+\ldots \quad (141),$$

where

$$c_4=\Delta_1.K^{\frac{1}{2}}c'^{\frac{1}{4}}\Lambda^{\frac{1}{2}} \quad (142),$$

$$N_{4,2s}=\left(\frac{\pi}{K}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}c_4 \quad (143),$$

$$N_{4,2s+1}=\frac{r'K\Lambda}{\pi^2}\left(\frac{\pi}{K}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2.K^{\frac{1}{2}}c'^{\frac{1}{4}}\Lambda^{\frac{1}{2}} \quad (144).$$

$$\frac{1}{i}\vartheta_5 = c_5 x + c'_5 y - \frac{1}{3!}(C_{5,0},\ C_{5,1},\ C_{5,2},\ C_{5,3}\between x,\ y)^3 + \ldots$$

$$+\frac{(-1)^n}{2n+1!}(P_{5,0},\ P_{5,1},\ P_{5,2},\ \ldots,\ P_{5,s},\ \ldots,\ P_{5,2n+1}\between x,\ y)^{2n+1} + \ldots \qquad (145),$$

where

$$c_5 = \frac{1}{K}\Delta_1 . c^{\frac{1}{4}} c'^{\frac{1}{4}} K^{\frac{3}{2}} \Lambda^{\frac{1}{2}} \qquad (146),$$

$$c_5' = \frac{r'K\Lambda}{\pi}\Delta_2 . c^{\frac{1}{4}} c'^{\frac{1}{4}} K^{\frac{1}{2}} \left(\frac{\pi}{\Lambda}\right)^2 \frac{d\Lambda^{\frac{1}{2}}}{dq'} \qquad (147),$$

$$P_{5,2s} = \frac{1}{K}\left(\frac{\pi}{K}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} K c_5 \qquad (148),$$

$$P_{5,2s+1} = K\Lambda \left(\frac{\pi}{K}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} \frac{c_5'}{K\Lambda} \qquad (149).$$

$$\vartheta_6 = c_6 - \frac{1}{2!}(B_{6,0},\ B_{6,1},\ B_{6,2}\between x,\ y)^2 + \ldots$$

$$+\frac{(-1)^n}{2n!}(N_{6,0},\ N_{6,1},\ N_{6,2},\ \ldots,\ N_{6,s},\ \ldots,\ N_{6,2n}\between x,\ y)^{2n} + \ldots \qquad (150),$$

where

$$c_6 = \Delta_1 . K^{\frac{1}{2}} c'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \gamma^{\frac{1}{4}} \qquad (151),$$

$$N_{6,2s} = \left(\frac{\pi}{K}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} c_6 \qquad (152),$$

$$N_{6,2s+1} = \frac{r'K\Lambda}{\pi^2}\left(\frac{\pi}{K}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2(s+1)} \frac{d^{n+1}}{dp'^{n-s} dq'^{s+1}} \Delta_2 . K^{\frac{1}{2}} c'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \gamma^{\frac{1}{4}} \qquad (153).$$

$$\frac{1}{i}\vartheta_7 = c_7 x + c'_7 y - \frac{1}{3!}(C_{7,0},\ C_{7,1},\ C_{7,2},\ C_{7,3}\between x,\ y)^3 + \ldots$$

$$+\frac{(-1)^n}{2n+1!}(P_{7,0},\ P_{7,1},\ P_{7,2},\ \ldots,\ P_{7,s},\ \ldots,\ P_{7,2n+1}\between x,\ y)^{2n+1} + \ldots \qquad (154),$$

where

$$c_7 = \frac{1}{K}\Delta_1 . c^{\frac{1}{4}} c'^{\frac{1}{4}} K^{\frac{3}{2}} \gamma^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \qquad (155),$$

$$P_{7,2s} = \frac{1}{K}\left(\frac{\pi}{K}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} K c_7 \qquad (156),$$

$$c'_7 = \frac{r'K\Lambda}{\pi^2}\Delta_2 . c^{\frac{1}{4}} c'^{\frac{1}{4}} K^{\frac{1}{2}} \frac{\pi^2}{\Lambda^2} \frac{d . \gamma^{\frac{1}{4}} \Lambda^{\frac{1}{2}}}{dq'} \qquad (157),$$

$$P_{7,2s+1} = K\Lambda \left(\frac{\pi}{K}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} \frac{c_7'}{K\Lambda} \qquad (158).$$

$$\vartheta_8 = c_8 - \frac{1}{2!}(\mathrm{B}_{8,0},\ \mathrm{B}_{8,1},\ \mathrm{B}_{8,2})(x,\ y)^2 + \ldots$$

$$+\frac{(-1)^n}{2n1!}(\mathrm{N}_{8,0},\ \mathrm{N}_{8,1},\ \mathrm{N}_{8,2},\ \ldots,\ \mathrm{N}_{8,s},\ \ldots,\ \mathrm{N}_{8,2n})(x,\ y)^{2n} + \ldots \quad (159),$$

where

$$c_8 = \Delta_1 . \mathrm{K}^{\frac{1}{2}} \gamma'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \quad (160),$$

$$\mathrm{N}_{8,2s} = \left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} c_8 \quad (161),$$

$$\mathrm{N}_{8,2s+1} = \frac{r'\mathrm{K}\Lambda}{\pi^2} \left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2(s+1)} \frac{d^{n+1}}{dp'^{n-s} dq'^{s+1}} \Delta_2 . \mathrm{K}^{\frac{1}{2}} \gamma'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \quad (162).$$

$$\vartheta_9 = c_9 - \frac{1}{2!}(\mathrm{B}_{9,0},\ \mathrm{B}_{9,1},\ \mathrm{B}_{9,2})(x,\ y)^2 + \ldots$$

$$+\frac{(-1)^n}{2n!}(\mathrm{N}_{9,0},\ \mathrm{N}_{9,1},\ \mathrm{N}_{9,2},\ \ldots,\ \mathrm{N}_{9,s},\ \ldots,\ \mathrm{N}_{9,2n})(x,\ y)^{2n} + \ldots \quad (163),$$

where

$$c_9 = \Delta_1 . c^{\frac{1}{4}} \mathrm{K}^{\frac{1}{2}} \gamma'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \quad (164),$$

$$\mathrm{N}_{9,2s} = \left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} c_9 \quad (165),$$

$$\mathrm{N}_{9,2s+1} = \frac{r'\mathrm{K}\Lambda}{\pi^2} \left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2(s+1)} \frac{d^{n+1}}{dp'^{n-s} dq'^{s+1}} \Delta_2 . c^{\frac{1}{4}} \mathrm{K}^{\frac{1}{2}} \gamma'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \quad (166).$$

$$\frac{1}{i}\vartheta_{10} = c_{10}x + c'_{10}y - \frac{1}{3!}(\mathrm{C}_{10,0},\ \mathrm{C}_{10,1},\ \mathrm{C}_{10,2},\ \mathrm{C}_{10,3})(x,\ y)^3 + \ldots$$

$$+\frac{(-1)^n}{2n+1!}(\mathrm{P}_{10,0},\ \mathrm{P}_{10,1},\ \mathrm{P}_{10,2},\ \ldots,\ \mathrm{P}_{10,s},\ \ldots,\ \mathrm{P}_{10,2n+1})(x,\ y)^{2n+1} + \ldots \quad (167),$$

where

$$c_{10} = \frac{r'\mathrm{K}\Lambda}{\pi^2} \Delta_2 . \gamma^{\frac{1}{4}} \gamma'^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \left(\frac{\pi}{\mathrm{K}}\right)^2 \frac{d\mathrm{K}^{\frac{1}{2}}}{dp'} \quad (168),$$

$$c'_{10} = \frac{1}{\Lambda} \Delta_1 . \gamma^{\frac{1}{4}} \gamma'^{\frac{1}{4}} \mathrm{K}^{\frac{1}{2}} \Lambda^{\frac{3}{2}} \quad (169),$$

$$\mathrm{P}_{10,2s} = \mathrm{K}\Lambda \left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} \frac{c_{10}}{\mathrm{K}\Lambda} \quad (170),$$

$$\mathrm{P}_{10,2s+1} = \frac{1}{\Lambda} \left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)} \left(\frac{\pi}{\Lambda}\right)^{2s} \frac{d^n}{dp'^{n-s} dq'^s} \Lambda c'_{10} \quad (171).$$

$$\frac{1}{i}\vartheta_{11}=c_{11}x+c'_{11}y-\frac{1}{3!}(\mathrm{C}_{11,0},\ \mathrm{C}_{11,1},\ \mathrm{C}_{11,2},\ \mathrm{C}_{11,3}\between x,\ y)^3+\ldots$$

$$+\frac{(-1)^n}{2n+1!}(\mathrm{P}_{11,0},\ \mathrm{P}_{11,1},\ \mathrm{P}_{11,2},\ \ldots,\ \mathrm{P}_{11,s},\ \ldots,\ \mathrm{P}_{11,2n+1}\between x,\ y)^{2n+1}+\ldots \quad (172),$$

where

$$c_{11}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\Delta_2.\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{1}{2}}\left(\frac{\pi}{\mathrm{K}}\right)^2\frac{d.c^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{dp'} \quad (173),$$

$$c'_{11}=\frac{1}{\Lambda}\Delta_1.\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}c^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{3}{2}} \quad (174),$$

$$\mathrm{P}_{11,2s}=\mathrm{K}\Lambda\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}\frac{c_{11}}{\mathrm{K}\Lambda} \quad (175),$$

$$\mathrm{P}_{11,2s+1}=\frac{1}{\Lambda}\left(\frac{\pi}{\Lambda}\right)^{2(n-s)}\left(\frac{\pi}{\mathrm{K}}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}\Lambda c'_{11} \quad (176).$$

$$\vartheta_{12}=c_{12}-\frac{1}{2!}(\mathrm{B}_{12,0},\ \mathrm{B}_{12,1},\ \mathrm{B}_{12,2}\between x,\ y)^2+\ldots$$

$$+\frac{(-1)^n}{2n!}(\mathrm{N}_{12,0},\ \mathrm{N}_{12,1},\ \mathrm{N}_{12,2},\ \ldots,\ \mathrm{N}_{12,s},\ \ldots,\ \mathrm{N}_{12,2n}\between x,\ y)^{2n}+\ldots \quad (177),$$

where

$$c_{12}=\Delta_1.c'^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{1}{2}}, \quad (178),$$

$$\mathrm{N}_{12,2s}=\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}c_{12} \quad (179),$$

$$\mathrm{N}_{12,2s+1}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2.c'^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{1}{2}} \quad (180).$$

$$\frac{1}{i}\vartheta_{13}=c_{13}x+c'_{13}y-\frac{1}{3!}(\mathrm{C}_{13,0},\ \mathrm{C}_{13,1},\ \mathrm{C}_{13,2},\ \mathrm{C}_{13,3}\between x,\ y)^3+\ldots$$

$$+\frac{(-1)^n}{2n+1!}(\mathrm{P}_{13,0},\ \mathrm{P}_{13,1},\ \mathrm{P}_{13,2},\ \ldots,\ \mathrm{P}_{13,s},\ \ldots,\ \mathrm{P}_{13,2n+1}\between x,\ y)^{2n+1}+\ldots \quad (181),$$

where

$$c_{13}=\frac{1}{\mathrm{K}}\Delta_1.c^{\frac{1}{4}}c'^{\frac{1}{4}}\mathrm{K}^{\frac{3}{2}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{1}{2}} \quad (182),$$

$$c'_{13}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\Delta_2.c^{\frac{1}{4}}c'^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}\left(\frac{\pi}{\Lambda}\right)^2\frac{d\gamma'^{\frac{1}{4}}\Lambda^{\frac{1}{2}}}{dq'} \quad (183),$$

$$\mathrm{P}_{13,2s}=\frac{1}{\mathrm{K}}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}\mathrm{K}c_{13} \quad (184),$$

$$\mathrm{P}_{13,2s+1}=\mathrm{K}\Lambda\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}\frac{c'_{13}}{\mathrm{K}\Lambda}. \quad (185).$$

$$\frac{1}{i}\vartheta_{14}=c_{14}x+c'_{14}y-\frac{1}{3!}(\mathrm{C}_{14,0},\ \mathrm{C}_{14,1},\ \mathrm{C}_{14,2},\ \mathrm{C}_{14,3})(x,\ y)^3+\ldots$$

$$+\frac{(-1)^n}{2n+1!}(\mathrm{P}_{14,0},\ \mathrm{P}_{14,1},\ \mathrm{P}_{14,2},\ \ldots,\ \mathrm{P}_{14,s},\ \ldots,\ \mathrm{P}_{14,2n+1})(x,\ y)^{2n+1}+\ldots \quad (186),$$

where

$$c_{14}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\Delta_2.\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{1}{2}}\left(\frac{\pi}{\mathrm{K}}\right)^2\frac{d.c'^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{dp'} \quad (187),$$

$$c'_{14}=\frac{1}{\Lambda}\Delta_1.c^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}\Lambda^{\frac{3}{2}} \quad (188),$$

$$\mathrm{P}_{14,2s}=\mathrm{K}\Lambda\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}\frac{c_{14}}{\mathrm{K}\Lambda} \quad (189),$$

$$\mathrm{P}_{14,2s+1}=\frac{1}{\Lambda}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}\Lambda c'_{14} \quad (190).$$

Lastly

$$\vartheta_{15}=c_{15}-\frac{1}{2!}(\mathrm{B}_{15,0},\ \mathrm{B}_{15,1},\ \mathrm{B}_{15,2})(x,\ y)^2$$

$$+\frac{(-1)^n}{2n!}(\mathrm{N}_{15,0},\ \mathrm{N}_{15,1},\ \mathrm{N}_{15,2},\ \ldots,\ \mathrm{N}_{15,s},\ \ldots,\ \mathrm{N}_{15,2n})(x,\ y)^{2n}+\ldots \quad (191),$$

where

$$c_{15}=\frac{r'}{\pi^2}\Delta_2.c^{\frac{1}{4}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}\mathrm{K}^{\frac{3}{2}}\Lambda^{\frac{3}{2}} \quad (192),$$

$$\mathrm{N}_{15,2s}=\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^s}c_{15} \quad (193),$$

$$\mathrm{N}_{15,2s+1}=\frac{1}{\mathrm{K}\Lambda}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s-1)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^{n-1}}{dp'^{n-s-1}dq'^s}\Delta_1.c^{\frac{1}{4}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}} \quad (194).$$

37. The formulæ (42), (43), (44) give expressions for κ_1, κ_2, κ_3 in terms of the c's; and therefore, by the preceding, all the κ's of § 13 can be expressed in terms of K, Λ. In fact, we have

$$\left.\begin{aligned}\kappa_1&=\frac{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}}\gamma^{\frac{1}{4}})}{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}})}\\ \kappa'_1&=\frac{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}})}{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Lambda_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}})}\end{aligned}\right\} \quad (195)$$

$$\left.\begin{aligned}\kappa_2&=\frac{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}}\gamma^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}}}\\ \kappa'_2&=\frac{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Delta^{\frac{1}{2}}\gamma^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}})}\end{aligned}\right\} \quad (196)$$

$$\left.\begin{aligned}\kappa_3&=\frac{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}})}\\ \kappa'_3&=\frac{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Delta_1.\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}})}\end{aligned}\right\} \quad (197)$$

$$K_3=\frac{r'}{\pi^2}\frac{(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}}\gamma^{\frac{1}{4}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}})\times(\Delta_2 . K^{\frac{3}{2}}\Lambda^{\frac{3}{2}}c^{\frac{1}{4}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}})} \quad . \quad . \quad . \quad . \quad . \quad (198)$$

$$K_2=\frac{r'}{\pi^2}\frac{(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}})\times(\Delta_2 . K^{\frac{3}{2}}\Lambda^{\frac{3}{2}}c^{\frac{1}{4}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}})} \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

$$K_1=\frac{r'}{\pi^2}\frac{(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c^{\frac{1}{4}}\gamma'^{\frac{1}{4}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}c'^{\frac{1}{4}}\gamma'^{\frac{1}{4}})\times(\Delta_2 . K^{\frac{3}{2}}\Lambda^{\frac{3}{2}}c^{\frac{1}{4}}c'^{\frac{1}{4}}\gamma^{\frac{1}{4}}\gamma'^{\frac{1}{4}})}{(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma^{\frac{1}{4}})\times(\Delta_1 . K^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\gamma'^{\frac{1}{4}})} . \quad . \quad . \quad . \quad . \quad (200).$$

SECTION III.

The Addition Theorem.

38. There is no addition theorem proper for the theta-functions, but the product of a theta-function of the sum of two variables by a theta-function of the difference of those variables can be expressed in terms of the functions of those variables. Thus with the previous notation for the single theta-functions

$$\theta_{0,1}{}^2(0)\theta_{0,1}(u+v)\theta_{0,1}(u-v)=\theta_{0,1}{}^2(u)\theta_{0,1}{}^2(v)-\theta_{1,1}{}^2(u)\theta_{1,1}{}^2(v), \quad . \quad . \quad . \quad . \quad . \quad (201),$$

$$\theta_{0,1}{}^2(0)\theta_{1,1}(u+v)\theta_{1,1}(u-v)=\theta_{1,1}{}^2(u)\theta_{0,1}{}^2(v)-\theta_{1,1}{}^2(v)\theta_{0,1}{}^2(u), \quad . \quad . \quad . \quad . \quad . \quad (202),$$

$$\theta_{0,0}(0)\theta_{0,1}(0)\theta_{1,1}(u+v)\theta_{0,1}(u-v)=\theta_{0,1}(u)\theta_{1,1}(u)\theta_{1,0}(v)\theta_{0,0}(v) + \theta_{0,1}(v)\theta_{1,1}(v)\theta_{1,0}(u)\theta_{0,0}(u) \quad . \quad . \quad (203).$$

$$\frac{\theta_{1,0}{}^2(0)}{\theta_{0,0}{}^2(0)}=\kappa, \quad \text{and} \quad \frac{\theta_{0,1}{}^2(0)}{\theta_{0,0}{}^2(0)}=\kappa'.$$

Dividing the third of these by the first and substituting for the θ's, there results the ordinary expression for $sn(u+v)$; and the division of the second by the first gives $sn(u+v)sn(u-v)$. The object of the present section is to obtain, by means of the theorem (23), the complete expression of the sum-and-difference theorem for the double theta-functions; it will be given by 256 formulæ similar to (201), (202), (203).

39. Some abbreviations in the notation are desirable; in the subsequent formulæ

$$\left.\begin{array}{lll}\Theta & \text{denotes} & \vartheta(x+\xi, y+\eta), \\ \Theta' & \text{,,} & \vartheta(x-\xi, y-\eta), \\ \vartheta & \text{,,} & \vartheta(x, y), \\ \theta & \text{,,} & \vartheta(\xi, \eta),\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204),$$

and, in order to simplify the first forms which are obtained from (23), some subsidiary equations are necessary. For example, writing down equations similar to (i)–(x) in Section I., but involving $\theta^2\vartheta^2$ instead of $c^2\vartheta^2$, the following simpler relations are found to be their equivalent and include (31), (32), (33) as particular cases.

$$\left.\begin{aligned}
\theta_0^2\vartheta_0^2+\theta_5^2\vartheta_5^2+\theta_{14}^2\vartheta_{14}^2&=\theta_1^2\vartheta_1^2+\theta_6^2\vartheta_6^2+\theta_{12}^2\vartheta_{12}^2\\
\theta_0^2\vartheta_0^2+\theta_{10}^2\vartheta_{10}^2+\theta_{13}^2\vartheta_{13}^2&=\theta_2^2\vartheta_2^2+\theta_9^2\vartheta_9^2+\theta_{12}^2\vartheta_{12}^2\\
\theta_0^2\vartheta_0^2+\theta_7^2\vartheta_7^2+\theta_{10}^2\vartheta_{10}^2&=\theta_3^2\vartheta_3^2+\theta_6^2\vartheta_6^2+\theta_8^2\vartheta_8^2\\
\theta_0^2\vartheta_0^2+\theta_5^2\vartheta_5^2+\theta_{11}^2\vartheta_{11}^2&=\theta_3^2\vartheta_3^2+\theta_4^2\vartheta_4^2+\theta_9^2\vartheta_9^2\\
\theta_0^2\vartheta_0^2+\theta_{11}^2\vartheta_{11}^2+\theta_{14}^2\vartheta_{14}^2&=\theta_2^2\vartheta_2^2+\theta_8^2\vartheta_8^2+\theta_{15}^2\vartheta_{15}^2\\
\theta_0^2\vartheta_0^2+\theta_7^2\vartheta_7^2+\theta_{13}^2\vartheta_{13}^2&=\theta_1^2\vartheta_1^2+\theta_4^2\vartheta_4^2+\theta_{15}^2\vartheta_{15}^2\\
\theta_1^2\vartheta_1^2+\theta_6^2\vartheta_6^2+\theta_{11}^2\vartheta_{11}^2&=\theta_2^2\vartheta_2^2+\theta_7^2\vartheta_7^2+\theta_9^2\vartheta_9^2\\
\theta_1^2\vartheta_1^2+\theta_4^2\vartheta_4^2+\theta_{10}^2\vartheta_{10}^2&=\theta_2^2\vartheta_2^2+\theta_5^2\vartheta_5^2+\theta_8^2\vartheta_8^2\\
\theta_4^2\vartheta_4^2+\theta_9^2\vartheta_9^2+\theta_{14}^2\vartheta_{14}^2&=\theta_6^2\vartheta_6^2+\theta_8^2\vartheta_8^2+\theta_{13}^2\vartheta_{13}^2\\
\theta_5^2\vartheta_5^2+\theta_{11}^2\vartheta_{11}^2+\theta_{14}^2\vartheta_{14}^2&=\theta_7^2\vartheta_7^2+\theta_{10}^2\vartheta_{10}^2+\theta_{13}^2\vartheta_{13}^2
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (205)$$

Then by (23) we have, for instance,

$$\begin{aligned}
4c_{12}^2\Theta_{12}\Theta'_{12}&=\theta_0^2\vartheta_0^2+\theta_4^2\vartheta_4^2+\theta_8^2\vartheta_8^2+\theta_{12}^2\vartheta_{12}^2+\theta_3^2\vartheta_3^2+\theta_7^2\vartheta_7^2+\theta_{11}^2\vartheta_{11}^2+\theta_{15}^2\vartheta_{15}^2\\
&\quad-\theta_1^2\vartheta_1^2-\theta_5^2\vartheta_5^2-\theta_9^2\vartheta_9^2-\theta_{13}^2\vartheta_{13}^2-\theta_2^2\vartheta_2^2-\theta_6^2\vartheta_6^2-\theta_{10}^2\vartheta_{10}^2-\theta_{14}^2\vartheta_{14}^2\\
&=4(\theta_{12}^2\vartheta_{12}^2+\theta_{11}^2\vartheta_{11}^2-\theta_{10}^2\vartheta_{10}^2-\theta_{13}^2\vartheta_{13}^2)
\end{aligned}$$

by equations (205) and therefore

$$c_{12}^2\Theta_{12}\Theta'_{12}=\theta_{12}^2\vartheta_{12}^2+\theta_{11}^2\vartheta_{11}^2-\theta_{10}^2\vartheta_{10}^2-\theta_{13}^2\vartheta_{13}^2$$

which reduces to an identity when ξ, η are both zero. Another similar set is

$$\left.\begin{aligned}
\theta_0^2\vartheta_5^2+\theta_2^2\vartheta_7^2+\theta_3^2\vartheta_6^2+\theta_1^2\vartheta_4^2&=\theta_5^2\vartheta_0^2+\theta_7^2\vartheta_2^2+\theta_6^2\vartheta_3^2+\theta_4^2\vartheta_1^2\\
\theta_0^2\vartheta_5^2+\theta_7^2\vartheta_2^2+\theta_9^2\vartheta_{12}^2+\theta_{14}^2\vartheta_{11}^2&=\theta_5^2\vartheta_0^2+\theta_2^2\vartheta_7^2+\theta_{12}^2\vartheta_9^2+\theta_{11}^2\vartheta_{14}^2\\
\theta_8^2\vartheta_{13}^2+\theta_{10}^2\vartheta_{15}^2+\theta_9^2\vartheta_{12}^2+\theta_{11}^2\vartheta_{14}^2&=\theta_{13}^2\vartheta_8^2+\theta_{15}^2\vartheta_{10}^2+\theta_{12}^2\vartheta_9^2+\theta_{14}^2\vartheta_{11}^2\\
\theta_8^2\vartheta_{13}^2+\theta_{15}^2\vartheta_{10}^2+\theta_1^2\vartheta_4^2+\theta_6^2\vartheta_3^2&=\theta_{13}^2\vartheta_8^2+\theta_{10}^2\vartheta_{15}^2+\theta_4^2\vartheta_1^2+\theta_3^2\vartheta_6^2\\
\theta_0^2\vartheta_5^2+\theta_{13}^2\vartheta_8^2+\theta_3^2\vartheta_6^2+\theta_{14}^2\vartheta_{11}^2&=\theta_5^2\vartheta_0^2+\theta_8^2\vartheta_{13}^2+\theta_6^2\vartheta_3^2+\theta_{11}^2\vartheta_{14}^2\\
\theta_6^2\vartheta_3^2+\theta_{14}^2\vartheta_{11}^2+\theta_7^2\vartheta_2^2+\theta_{15}^2\vartheta_{10}^2&=\theta_3^2\vartheta_6^2+\theta_{11}^2\vartheta_{14}^2+\theta_2^2\vartheta_7^2+\theta_{10}^2\vartheta_{15}^2\\
\theta_2^2\vartheta_7^2+\theta_{15}^2\vartheta_{10}^2+\theta_1^2\vartheta_4^2+\theta_{12}^2\vartheta_9^2&=\theta_7^2\vartheta_2^2+\theta_{10}^2\vartheta_{15}^2+\theta_4^2\vartheta_1^2+\theta_9^2\vartheta_{12}^2\\
\theta_{12}^2\vartheta_9^2+\theta_4^2\vartheta_1^2+\theta_{13}^2\vartheta_8^2+\theta_5^2\vartheta_0^2&=\theta_9^2\vartheta_{12}^2+\theta_1^2\vartheta_4^2+\theta_8^2\vartheta_{13}^2+\theta_0^2\vartheta_5^2\\
\theta_{12}^2\vartheta_9^2+\theta_6^2\vartheta_3^2+\theta_5^2\vartheta_0^2+\theta_{15}^2\vartheta_{10}^2&=\theta_9^2\vartheta_{12}^2+\theta_3^2\vartheta_6^2+\theta_0^2\vartheta_5^2+\theta_{10}^2\vartheta_{15}^2\\
\theta_{12}^2\vartheta_9^2+\theta_3^2\vartheta_6^2+\theta_{13}^2\vartheta_8^2+\theta_2^2\vartheta_7^2&=\theta_9^2\vartheta_{12}^2+\theta_6^2\vartheta_3^2+\theta_8^2\vartheta_{13}^2+\theta_7^2\vartheta_2^2\\
\theta_8^2\vartheta_{13}^2+\theta_2^2\vartheta_7^2+\theta_1^2\vartheta_4^2+\theta_{11}^2\vartheta_{14}^2&=\theta_{13}^2\vartheta_8^2+\theta_7^2\vartheta_2^2+\theta_4^2\vartheta_1^2+\theta_{14}^2\vartheta_{11}^2\\
\theta_0^2\vartheta_5^2+\theta_{15}^2\vartheta_{10}^2+\theta_1^2\vartheta_4^2+\theta_{14}^2\vartheta_{11}^2&=\theta_5^2\vartheta_0^2+\theta_{10}^2\vartheta_{15}^2+\theta_4^2\vartheta_1^2+\theta_{11}^2\vartheta_{14}^2
\end{aligned}\right\}\quad (206)$$

and by means of these we obtain, among others, the result

$$c_8^2\Theta_{13}\Theta'_{13}=\theta_8^2\vartheta_{13}^2-\theta_{13}^2\vartheta_8^2+\theta_{11}^2\vartheta_{14}^2-\theta_{14}^2\vartheta_{11}^2,$$

which as in the previous case reduces to an identity when ξ, η are both zero. A set similar to the last, and likewise necessary, is

$$\left.\begin{aligned}
\theta_8^2\vartheta_2^2+\theta_{12}^2\vartheta_6^2+\theta_{14}^2\vartheta_4^2+\theta_{10}^2\vartheta_0^2&=\theta_2^2\vartheta_8^2+\theta_6^2\vartheta_{12}^2+\theta_4^2\vartheta_{14}^2+\theta_0^2\vartheta_{10}^2\\
\theta_8^2\vartheta_2^2+\theta_6^2\vartheta_{12}^2+\theta_{11}^2\vartheta_1^2+\theta_5^2\vartheta_{15}^2&=\theta_2^2\vartheta_8^2+\theta_{12}^2\vartheta_6^2+\theta_1^2\vartheta_{11}^2+\theta_{15}^2\vartheta_5^2\\
\theta_1^2\vartheta_{11}^2+\theta_5^2\vartheta_{15}^2+\theta_7^2\vartheta_{13}^2+\theta_3^2\vartheta_9^2&=\theta_{11}^2\vartheta_1^2+\theta_{15}^2\vartheta_5^2+\theta_{13}^2\vartheta_7^2+\theta_9^2\vartheta_3^2\\
\theta_{13}^2\vartheta_7^2+\theta_3^2\vartheta_9^2+\theta_{14}^2\vartheta_4^2+\theta_0^2\vartheta_{10}^2&=\theta_7^2\vartheta_{13}^2+\theta_9^2\vartheta_3^2+\theta_4^2\vartheta_{14}^2+\theta_{10}^2\vartheta_0^2\\
\theta_1^2\vartheta_{11}^2+\theta_4^2\vartheta_{14}^2+\theta_2^2\vartheta_8^2+\theta_7^2\vartheta_{13}^2&=\theta_{11}^2\vartheta_1^2+\theta_{14}^2\vartheta_4^2+\theta_8^2\vartheta_2^2+\theta_{13}^2\vartheta_7^2\\
\theta_1^2\vartheta_{11}^2+\theta_{14}^2\vartheta_4^2+\theta_3^2\vartheta_9^2+\theta_{12}^2\vartheta_6^2&=\theta_{11}^2\vartheta_1^2+\theta_4^2\vartheta_{14}^2+\theta_9^2\vartheta_3^2+\theta_6^2\vartheta_{12}^2\\
\theta_{12}^2\vartheta_6^2+\theta_9^2\vartheta_3^2+\theta_{10}^2\vartheta_0^2+\theta_{15}^2\vartheta_5^2&=\theta_6^2\vartheta_{12}^2+\theta_3^2\vartheta_9^2+\theta_0^2\vartheta_{10}^2+\theta_5^2\vartheta_{15}^2\\
\theta_0^2\vartheta_{10}^2+\theta_{15}^2\vartheta_5^2+\theta_2^2\vartheta_8^2+\theta_{13}^2\vartheta_7^2&=\theta_{10}^2\vartheta_0^2+\theta_5^2\vartheta_{15}^2+\theta_8^2\vartheta_2^2+\theta_7^2\vartheta_{13}^2\\
\theta_1^2\vartheta_{11}^2+\theta_0^2\vartheta_{10}^2+\theta_2^2\vartheta_8^2+\theta_3^2\theta_9^2&=\theta_{11}^2\vartheta_1^2+\theta_{10}^2\vartheta_0^2+\theta_8^2\vartheta_2^2+\theta_9^2\vartheta_3^2\\
\theta_1^2\vartheta_{11}^2+\theta_{10}^2\vartheta_0^2+\theta_7^2\vartheta_{13}^2+\theta_{12}^2\vartheta_6^2&=\theta_{11}^2\vartheta_1^2+\theta_0^2\vartheta_{10}^2+\theta_{13}^2\vartheta_7^2+\theta_6^2\vartheta_{12}^2\\
\theta_{12}^2\vartheta_6^2+\theta_{13}^2\vartheta_7^2+\theta_{14}^2\vartheta_4^2+\theta_{15}^2\vartheta_5^2&=\theta_6^2\vartheta_{12}^2+\theta_7^2\vartheta_{13}^2+\theta_4^2\vartheta_{14}^2+\theta_5^2\vartheta_{15}^2\\
\theta_8^2\vartheta_2^2+\theta_3^2\vartheta_9^2+\theta_{14}^2\vartheta_4^2+\theta_5^2\vartheta_{15}^2&=\theta_2^2\vartheta_8^2+\theta_9^2\vartheta_3^2+\theta_4^2\vartheta_{14}^2+\theta_{15}^2\vartheta_5^2
\end{aligned}\right\} . \quad (207).$$

The equations necessary to reduce the forms first given for such expressions as $c_2c_3\Theta_1\Theta'_0$ will be found at the end of the sixteen sets, each of sixteen equations.

40. First Set, with Θ'_0.

$$\begin{aligned}
c_0c_0\,\Theta_0\,\Theta'_0&=\theta_0\theta_0\,\vartheta_0\,\vartheta_0+\theta_7\,\theta_7\,\vartheta_7\,\vartheta_7+\theta_{10}\theta_{10}\vartheta_{10}\vartheta_{10}+\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}\\
c_2c_3\,\Theta_1\,\Theta'_0&=\theta_2\theta_3\,\vartheta_1\,\vartheta_0+\theta_6\,\theta_7\,\vartheta_4\,\vartheta_5-\theta_{10}\theta_{11}\vartheta_8\,\vartheta_9-\theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}\\
c_1c_3\,\Theta_2\,\Theta'_0&=\theta_1\theta_3\,\vartheta_2\,\vartheta_0-\theta_5\,\theta_7\,\vartheta_4\,\vartheta_6+\theta_9\,\theta_{11}\vartheta_8\,\vartheta_{10}-\theta_{13}\theta_{15}\vartheta_{12}\vartheta_{14}\\
c_1c_2\,\Theta_3\,\Theta'_0&=\theta_1\theta_2\,\vartheta_3\,\vartheta_0+\theta_4\,\theta_7\,\vartheta_5\,\vartheta_6+\theta_8\,\theta_{11}\vartheta_9\,\vartheta_{10}+\theta_{13}\theta_{14}\vartheta_{12}\vartheta_{15}\\
c_8c_{12}\Theta_4\,\Theta'_0&=\theta_8\theta_{12}\vartheta_4\,\vartheta_0+\theta_{10}\theta_{14}\vartheta_2\,\vartheta_6-\theta_9\,\theta_{13}\vartheta_1\,\vartheta_5-\theta_{11}\theta_{15}\vartheta_3\,\vartheta_7\\
c_1c_4\,\Theta_5\,\Theta'_0&=\theta_1\theta_4\,\vartheta_5\,\vartheta_0+\theta_0\,\theta_5\,\vartheta_1\,\vartheta_4+\theta_{10}\theta_{15}\vartheta_{11}\vartheta_{14}+\theta_{14}\theta_{11}\vartheta_{10}\vartheta_{15}\\
c_9c_{15}\Theta_6\,\Theta'_0&=\theta_9\theta_{15}\vartheta_6\,\vartheta_0+\theta_1\,\theta_7\,\vartheta_8\,\vartheta_{14}-\theta_{13}\theta_{11}\vartheta_4\,\vartheta_2-\theta_3\,\theta_5\,\vartheta_{10}\vartheta_{12}\\
c_1c_6\,\Theta_7\,\Theta'_0&=\theta_1\theta_6\,\vartheta_7\,\vartheta_0+\theta_0\,\theta_7\,\vartheta_1\,\vartheta_6+\theta_{10}\theta_{13}\vartheta_{12}\vartheta_{11}+\theta_{11}\theta_{12}\vartheta_{10}\vartheta_{13}\\
c_4c_{12}\Theta_8\,\Theta'_0&=\theta_4\theta_{12}\vartheta_8\,\vartheta_0+\theta_5\,\theta_{13}\vartheta_9\,\vartheta_1-\theta_6\,\theta_{14}\vartheta_2\,\vartheta_{10}-\theta_7\,\theta_{15}\vartheta_3\,\vartheta_{11}\\
c_6c_{15}\Theta_9\,\Theta'_0&=\theta_6\theta_{15}\vartheta_9\,\vartheta_0-\theta_{14}\theta_7\,\vartheta_8\,\vartheta_1+\theta_2\,\theta_{11}\vartheta_4\,\vartheta_{13}-\theta_3\,\theta_{10}\vartheta_{12}\vartheta_5\\
c_2c_8\,\Theta_{10}\Theta'_0&=\theta_2\theta_8\,\vartheta_{10}\vartheta_0+\theta_{10}\theta_0\,\vartheta_8\,\vartheta_2+\theta_{13}\theta_7\,\vartheta_5\,\vartheta_{15}+\theta_5\,\theta_{15}\vartheta_7\,\vartheta_{13}\\
c_3c_8\,\Theta_{11}\Theta'_0&=\theta_3\theta_8\,\vartheta_{11}\vartheta_0+\theta_{11}\theta_0\,\vartheta_8\,\vartheta_3+\theta_5\,\theta_{14}\vartheta_6\,\vartheta_{13}+\theta_6\,\theta_{13}\vartheta_5\,\vartheta_{14}\\
c_4c_8\,\Theta_{12}\Theta'_0&=\theta_4\theta_8\,\vartheta_{12}\vartheta_0-\theta_1\,\theta_{13}\vartheta_5\,\vartheta_9+\theta_6\,\theta_{10}\vartheta_2\,\vartheta_{14}-\theta_7\,\theta_{11}\vartheta_3\,\vartheta_{15}\\
c_1c_{12}\Theta_{13}\Theta'_0&=\theta_1\theta_{12}\vartheta_{13}\vartheta_0+\theta_{13}\theta_0\,\vartheta_1\,\vartheta_{12}-\theta_7\,\theta_{10}\vartheta_6\,\vartheta_{11}-\theta_6\,\theta_{11}\vartheta_7\,\vartheta_{10}\\
c_2c_{12}\Theta_{14}\Theta'_0&=\theta_2\theta_{12}\vartheta_{14}\vartheta_0+\theta_{14}\theta_0\,\vartheta_2\,\vartheta_{12}-\theta_5\,\theta_{11}\vartheta_7\,\vartheta_9-\theta_7\,\theta_9\,\vartheta_5\,\vartheta_{11}\\
c_6c_9\,\Theta_{15}\Theta'_0&=\theta_6\theta_9\,\vartheta_{15}\vartheta_0+\theta_5\,\theta_{10}\vartheta_{12}\vartheta_3+\theta_4\,\theta_{11}\vartheta_2\,\vartheta_{13}+\theta_7\,\theta_8\,\vartheta_1\,\vartheta_{14}
\end{aligned}$$

Königsberger, in his paper already referred to, gives a set of sixteen similar equations expressing the sixteen functions $\Theta_r\Theta'_0$ in terms of products to which the above are similar; but the constants on the left-hand side are for the most part different from his, with the result of making the combinations on the right-hand side different.

41. SECOND Set, with Θ'_1.

$$c_2c_3\,\Theta_0\,\Theta'_1=\theta_2\theta_3\,\vartheta_0\,\vartheta_1-\theta_6\,\theta_7\,\vartheta_4\vartheta_5-\theta_{10}\theta_{11}\vartheta_8\,\vartheta_9+\theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}$$
$$c_1c_1\,\Theta_1\,\Theta'_1=\theta_1\theta_1\,\vartheta_1\,\vartheta_1-\theta_7\,\theta_7\,\vartheta_7\vartheta_7+\theta_{11}\theta_{11}\vartheta_{11}\vartheta_{11}-\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}$$
$$c_0c_3\,\Theta_2\,\Theta'_1=\theta_0\theta_3\,\vartheta_2\,\vartheta_1-\theta_5\,\theta_6\,\vartheta_4\vartheta_7+\theta_8\,\theta_{11}\vartheta_9\,\vartheta_{10}-\theta_{13}\theta_{14}\vartheta_{12}\vartheta_{15}$$
$$c_0c_2\,\Theta_3\,\Theta'_1=\theta_0\theta_2\,\vartheta_3\,\vartheta_1+\theta_4\,\theta_6\,\vartheta_5\vartheta_7+\theta_8\,\theta_{10}\vartheta_9\,\vartheta_{11}+\theta_{13}\theta_{15}\vartheta_{12}\vartheta_{14}$$
$$c_3c_6\,\Theta_4\,\Theta'_1=\theta_3\theta_6\,\vartheta_4\,\vartheta_1-\theta_2\,\theta_7\,\vartheta_5\vartheta_0-\theta_{11}\theta_{14}\vartheta_9\,\vartheta_{12}+\theta_{10}\theta_{15}\vartheta_8\,\vartheta_{13}$$
$$c_2c_6\,\Theta_5\,\Theta'_1=\theta_2\theta_6\,\vartheta_5\,\vartheta_1+\theta_1\,\theta_5\,\vartheta_2\vartheta_6-\theta_{10}\theta_{14}\vartheta_9\,\vartheta_{13}-\theta_9\,\theta_{13}\vartheta_{10}\vartheta_{14}$$
$$c_4c_3\,\Theta_6\,\Theta'_1=\theta_4\theta_3\,\vartheta_6\,\vartheta_1-\theta_2\,\theta_5\,\vartheta_7\vartheta_0+\theta_{11}\theta_{12}\vartheta_9\,\vartheta_{14}-\theta_{13}\theta_{10}\vartheta_8\,\vartheta_{15}$$
$$c_0c_6\,\Theta_7\,\Theta'_1=\theta_0\theta_6\,\vartheta_7\,\vartheta_1+\theta_7\,\theta_1\,\vartheta_0\vartheta_6+\theta_{11}\theta_{13}\vartheta_{10}\vartheta_{12}+\theta_{10}\theta_{12}\vartheta_{11}\vartheta_{13}$$
$$c_0c_9\,\Theta_8\,\Theta'_1=\theta_0\theta_9\,\vartheta_8\,\vartheta_1-\theta_3\,\theta_{10}\vartheta_2\vartheta_{11}-\theta_4\,\theta_{13}\vartheta_5\,\vartheta_{12}+\theta_7\,\theta_{14}\vartheta_6\,\vartheta_{15}$$
$$c_0c_8\,\Theta_9\,\Theta'_1=\theta_0\theta_8\,\vartheta_9\,\vartheta_1-\theta_2\,\theta_{10}\vartheta_3\vartheta_{11}+\theta_5\,\theta_{13}\vartheta_4\,\vartheta_{12}-\theta_7\,\theta_{15}\vartheta_6\,\vartheta_{14}$$
$$c_3c_8\,\Theta_{10}\Theta'_1=\theta_3\theta_8\,\vartheta_{10}\vartheta_1+\theta_{10}\theta_1\,\vartheta_3\vartheta_8-\theta_7\,\theta_{12}\vartheta_5\,\vartheta_{14}-\theta_5\,\theta_{14}\vartheta_7\,\vartheta_{12}$$
$$c_3c_9\,\Theta_{11}\Theta'_1=\theta_3\theta_9\,\vartheta_{11}\vartheta_1+\theta_{11}\theta_1\,\vartheta_3\vartheta_9-\theta_5\,\theta_{15}\vartheta_7\,\vartheta_{13}-\theta_7\,\theta_{13}\vartheta_5\,\vartheta_{15}$$
$$c_4c_9\,\Theta_{12}\Theta'_1=\theta_4\theta_9\,\vartheta_{12}\vartheta_1-\theta_0\,\theta_{13}\vartheta_8\vartheta_5-\theta_3\,\theta_{14}\vartheta_6\,\vartheta_{11}+\theta_7\,\theta_{10}\vartheta_2\,\vartheta_{15}$$
$$c_0c_{12}\Theta_{13}\Theta'_1=\theta_0\theta_{12}\vartheta_{13}\vartheta_1+\theta_1\,\theta_{13}\vartheta_0\vartheta_{12}-\theta_7\,\theta_{11}\vartheta_6\,\vartheta_{10}-\theta_6\,\theta_{10}\vartheta_7\,\vartheta_{11}$$
$$c_0c_{15}\Theta_{14}\Theta'_1=\theta_0\theta_{15}\vartheta_{14}\vartheta_1+\theta_{14}\theta_1\,\vartheta_0\vartheta_{15}-\theta_5\,\theta_{10}\vartheta_4\,\vartheta_{11}-\theta_4\,\theta_{11}\vartheta_5\,\vartheta_{10}$$
$$c_6c_8\,\Theta_{15}\Theta'_1=\theta_6\theta_8\,\vartheta_{15}\vartheta_1+\theta_5\,\theta_{11}\vartheta_2\vartheta_{12}+\theta_0\,\theta_{14}\vartheta_9\,\vartheta_7+\theta_3\,\theta_{13}\vartheta_4\,\vartheta_{10}$$

42. THIRD Set, with Θ'_2.

$$c_1c_3\,\Theta_0\,\Theta'_2=\theta_1\theta_3\,\vartheta_0\,\vartheta_2-\theta_5\,\theta_7\,\vartheta_4\vartheta_6-\theta_9\,\theta_{11}\vartheta_8\,\vartheta_{10}+\theta_{13}\theta_{15}\vartheta_{12}\vartheta_{14}$$
$$c_0c_3\,\Theta_1\,\Theta'_2=\theta_0\theta_3\,\vartheta_1\,\vartheta_2+\theta_5\,\theta_6\,\vartheta_4\vartheta_7-\theta_8\,\theta_{11}\vartheta_9\,\vartheta_{10}-\theta_{13}\theta_{14}\vartheta_{12}\vartheta_{15}$$
$$c_2c_2\,\Theta_2\,\Theta'_2=\theta_2\theta_2\,\vartheta_2\,\vartheta_2+\theta_5\,\theta_5\,\vartheta_5\vartheta_5-\theta_{10}\theta_{10}\vartheta_{10}\vartheta_{10}-\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}$$
$$c_0c_1\,\Theta_3\,\Theta'_2=\theta_0\theta_1\,\vartheta_3\,\vartheta_2+\theta_4\,\theta_5\,\vartheta_6\vartheta_7+\theta_{10}\theta_{11}\vartheta_8\,\vartheta_9+\theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}$$
$$c_0c_6\,\Theta_4\,\Theta'_2=\theta_0\theta_6\,\vartheta_4\,\vartheta_2-\theta_3\,\theta_5\,\vartheta_1\vartheta_7-\theta_{10}\theta_{12}\vartheta_8\,\vartheta_{14}+\theta_{11}\theta_{13}\vartheta_9\,\vartheta_{15}$$
$$c_1c_6\,\Theta_5\,\Theta'_2=\theta_1\theta_6\,\vartheta_5\,\vartheta_2+\theta_2\,\theta_5\,\vartheta_1\vartheta_6-\theta_{10}\theta_{13}\vartheta_9\,\vartheta_{14}-\theta_9\,\theta_{14}\vartheta_{10}\vartheta_{13}$$
$$c_0c_4\,\Theta_6\,\Theta'_2=\theta_0\theta_4\,\vartheta_6\,\vartheta_2-\theta_1\,\theta_5\,\vartheta_3\vartheta_7+\theta_{10}\theta_{14}\vartheta_8\,\vartheta_{12}-\theta_{11}\theta_{15}\vartheta_9\,\vartheta_{13}$$
$$c_1c_4\,\Theta_7\,\Theta'_2=\theta_1\theta_4\,\vartheta_7\,\vartheta_2+\theta_7\,\theta_2\,\vartheta_1\vartheta_4+\theta_{11}\theta_{14}\vartheta_8\,\vartheta_{13}+\theta_8\,\theta_{13}\vartheta_{11}\vartheta_{14}$$
$$c_3c_9\,\Theta_8\,\Theta'_2=\theta_3\theta_9\,\vartheta_8\,\vartheta_2-\theta_1\,\theta_{11}\vartheta_0\vartheta_{10}-\theta_7\,\theta_{13}\vartheta_{12}\vartheta_6+\theta_5\,\theta_{15}\vartheta_4\,\vartheta_{14}$$
$$c_3c_8\,\Theta_9\,\Theta'_2=\theta_3\theta_8\,\vartheta_9\,\vartheta_2-\theta_0\,\theta_{11}\vartheta_1\vartheta_{10}+\theta_6\,\theta_{13}\vartheta_7\,\vartheta_{12}-\theta_5\,\theta_{14}\vartheta_4\,\vartheta_{15}$$
$$c_0c_8\,\Theta_{10}\Theta'_2=\theta_0\theta_8\,\vartheta_{10}\vartheta_2+\theta_2\,\theta_{10}\vartheta_0\vartheta_8+\theta_7\,\theta_{15}\vartheta_5\,\vartheta_{13}+\theta_5\,\theta_{13}\vartheta_7\,\vartheta_{15}$$
$$c_0c_9\,\Theta_{11}\Theta'_2=\theta_0\theta_9\,\vartheta_{11}\vartheta_2+\theta_2\,\theta_{11}\vartheta_0\vartheta_9+\theta_7\,\theta_{14}\vartheta_5\,\vartheta_{12}+\theta_5\,\theta_{12}\vartheta_7\,\vartheta_{14}$$
$$c_6c_8\,\Theta_{12}\Theta'_2=\theta_6\theta_8\,\vartheta_{12}\vartheta_2-\theta_3\,\theta_{13}\vartheta_7\vartheta_9-\theta_0\,\theta_{14}\vartheta_4\,\vartheta_{10}+\theta_5\,\theta_{11}\vartheta_1\,\vartheta_{15}$$
$$c_3c_{12}\Theta_{13}\Theta'_2=\theta_3\theta_{12}\vartheta_{13}\vartheta_2+\theta_2\,\theta_{13}\vartheta_3\vartheta_{12}-\theta_5\,\theta_{10}\vartheta_4\,\vartheta_{11}-\theta_4\,\theta_{11}\vartheta_5\,\vartheta_{10}$$
$$c_0c_{12}\Theta_{14}\Theta'_2=\theta_0\theta_{12}\vartheta_{14}\vartheta_2+\theta_2\,\theta_{14}\vartheta_0\vartheta_{12}-\theta_{11}\theta_7\,\vartheta_5\,\vartheta_9-\theta_5\,\theta_9\,\vartheta_{11}\vartheta_7$$
$$c_1c_{12}\Theta_{15}\Theta'_2=\theta_1\theta_{12}\vartheta_{15}\vartheta_2+\theta_{11}\theta_6\,\vartheta_5\vartheta_8+\theta_7\,\theta_{10}\vartheta_4\,\vartheta_9+\theta_0\,\theta_{13}\vartheta_3\,\vartheta_{14}$$

43. Fourth Set—with Θ'_3.

$$c_1c_2\,\Theta_0\,\Theta'_3=\theta_1\theta_2\,\vartheta_0\,\vartheta_3-\theta_4\,\theta_7\,\vartheta_5\,\vartheta_6-\theta_8\,\theta_{11}\vartheta_9\,\vartheta_{10}+\theta_{13}\theta_{14}\vartheta_{12}\vartheta_{15}$$

$$c_0c_2\,\Theta_1\,\Theta'_3=\theta_0\theta_2\,\vartheta_1\,\vartheta_3+\theta_4\,\theta_6\,\vartheta_5\,\vartheta_7-\theta_8\,\theta_{10}\vartheta_9\,\vartheta_{11}-\theta_{13}\theta_{15}\vartheta_{12}\vartheta_{14}$$

$$c_0c_1\,\Theta_2\,\Theta'_3=\theta_0\theta_1\,\vartheta_2\,\vartheta_3-\theta_4\,\theta_5\,\vartheta_6\,\vartheta_7+\theta_{10}\theta_{11}\vartheta_8\,\vartheta_9-\theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}$$

$$c_3c_3\,\Theta_3\,\Theta'_3=\theta_3\theta_3\,\vartheta_3\,\vartheta_3-\theta_5\,\theta_5\,\vartheta_5\,\vartheta_5-\theta_{11}\theta_{11}\vartheta_{11}\vartheta_{11}+\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}$$

$$c_1c_6\,\Theta_4\,\Theta'_3=\theta_1\theta_6\,\vartheta_4\,\vartheta_3-\theta_0\,\theta_7\,\vartheta_2\,\vartheta_5-\theta_{11}\theta_{12}\vartheta_9\,\vartheta_{14}+\theta_{10}\theta_{13}\vartheta_8\,\vartheta_{15}$$

$$c_0c_6\,\Theta_5\,\Theta'_3=\theta_0\theta_6\,\vartheta_5\,\vartheta_3+\theta_0\,\theta_3\,\vartheta_5\,\vartheta_6-\theta_{11}\theta_{13}\vartheta_8\,\vartheta_{14}-\theta_8\,\theta_{14}\vartheta_{11}\vartheta_{13}$$

$$c_1c_4\,\Theta_6\,\Theta'_3=\theta_1\theta_4\,\vartheta_6\,\vartheta_3-\theta_0\,\theta_5\,\vartheta_2\,\vartheta_7+\theta_{11}\theta_{14}\vartheta_9\,\vartheta_{12}-\theta_{10}\theta_{15}\vartheta_8\,\vartheta_{13}$$

$$c_0c_4\,\Theta_7\,\Theta'_3=\theta_0\theta_4\,\vartheta_7\,\vartheta_3+\theta_7\,\theta_3\,\vartheta_0\,\vartheta_4+\theta_{10}\theta_{14}\vartheta_9\,\vartheta_{13}+\theta_9\,\theta_{13}\vartheta_{10}\vartheta_{14}$$

$$c_2c_9\,\Theta_8\,\Theta'_3=\theta_2\theta_9\,\vartheta_8\,\vartheta_3-\theta_0\,\theta_{11}\vartheta_1\,\vartheta_{10}-\theta_7\,\theta_{12}\vartheta_6\,\vartheta_{13}+\theta_5\,\theta_{14}\vartheta_4\,\vartheta_{15}$$

$$c_2c_8\,\Theta_9\,\Theta'_3=\theta_2\theta_8\,\vartheta_9\,\vartheta_3-\theta_0\,\theta_{10}\vartheta_1\,\vartheta_{11}+\theta_7\,\theta_{13}\vartheta_6\,\vartheta_{12}-\theta_4\,\theta_{14}\,\vartheta_5\vartheta_{15}$$

$$c_1c_8\,\Theta_{10}\Theta'_3=\theta_1\theta_8\,\vartheta_{10}\vartheta_3+\theta_{10}\theta_3\,\vartheta_1\,\vartheta_8-\theta_7\,\theta_{14}\vartheta_5\,\vartheta_{12}-\theta_5\,\theta_{12}\vartheta_7\,\vartheta_{14}$$

$$c_0c_8\,\Theta_{11}\Theta'_3=\theta_0\theta_8\,\vartheta_{11}\vartheta_3+\theta_{11}\theta_3\,\vartheta_0\,\vartheta_8+\theta_5\,\theta_{13}\vartheta_6\,\vartheta_{14}+\theta_6\,\theta_{14}\vartheta_5\,\vartheta_{13}$$

$$c_0c_{15}\Theta_{12}\Theta'_3=\theta_0\theta_{15}\vartheta_{12}\vartheta_3-\theta_5\,\theta_{10}\vartheta_6\,\vartheta_9-\theta_4\,\theta_{11}\vartheta_7\,\vartheta_8+\theta_6\,\theta_{14}\vartheta_2\,\vartheta_{13}$$

$$c_2c_{12}\Theta_{13}\Theta'_3=\theta_2\theta_{12}\vartheta_{13}\vartheta_3+\theta_3\,\theta_{13}\vartheta_2\,\vartheta_{12}-\theta_5\,\theta_{11}\vartheta_{10}\vartheta_4-\theta_4\,\theta_{10}\vartheta_5\,\vartheta_{11}$$

$$c_4c_9\,\Theta_{14}\Theta'_3=\theta_4\theta_9\,\vartheta_{14}\vartheta_3+\theta_3\,\theta_{14}\vartheta_4\,\vartheta_9-\theta_0\,\theta_{13}\vartheta_7\,\vartheta_{10}-\theta_7\,\theta_{10}\vartheta_0\,\vartheta_{13}$$

$$c_0c_{12}\Theta_{15}\Theta'_3=\theta_0\theta_{12}\vartheta_{15}\vartheta_3+\theta_9\,\theta_5\,\vartheta_{10}\vartheta_6+\theta_{11}\theta_7\,\vartheta_4\,\vartheta_8+\theta_2\,\theta_{14}\vartheta_1\,\vartheta_{13}$$

44. Fifth Set—with Θ'_4.

$$c_8c_{12}\Theta_0\,\Theta'_4=\theta_8\theta_{12}\vartheta_0\,\vartheta_4+\theta_{10}\theta_{14}\vartheta_2\,\vartheta_6+\theta_9\,\theta_{13}\vartheta_1\,\vartheta_5+\theta_{11}\theta_{15}\vartheta_3\,\vartheta_7$$

$$c_3c_6\,\Theta_1\,\Theta'_4=\theta_3\theta_6\,\vartheta_1\,\vartheta_4+\theta_2\,\theta_7\,\vartheta_0\,\vartheta_5-\theta_{11}\theta_{14}\vartheta_9\,\vartheta_{12}-\theta_{10}\theta_{15}\vartheta_8\,\vartheta_{13}$$

$$c_0c_6\,\Theta_2\,\Theta'_4=\theta_0\theta_6\,\vartheta_2\,\vartheta_4+\theta_3\,\theta_5\,\vartheta_1\,\vartheta_7+\theta_{10}\theta_{12}\vartheta_8\,\vartheta_{14}+\theta_{11}\theta_{13}\vartheta_9\,\vartheta_{15}$$

$$c_1c_6\,\Theta_3\,\Theta'_4=\theta_1\theta_6\,\vartheta_3\,\vartheta_4+\theta_0\,\theta_7\,\vartheta_2\,\vartheta_5+\theta_{11}\theta_{12}\vartheta_9\,\vartheta_{14}+\theta_{10}\theta_{13}\vartheta_8\,\vartheta_{15}$$

$$c_4c_4\,\Theta_4\,\Theta'_4=\theta_4\theta_4\,\vartheta_4\,\vartheta_4-\theta_7\,\theta_7\,\vartheta_7\,\vartheta_7-\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}+\theta_{14}\theta_{14}\vartheta_{14}\vartheta_{14}$$

$$c_0c_1\,\Theta_5\,\Theta'_4=\theta_0\theta_1\,\vartheta_5\,\vartheta_4+\theta_5\,\theta_4\,\vartheta_0\,\vartheta_1+\theta_{14}\theta_{15}\vartheta_{10}\vartheta_{11}+\theta_{10}\theta_{11}\vartheta_{14}\vartheta_{15}$$

$$c_1c_3\,\Theta_6\,\Theta'_4=\theta_1\theta_3\,\vartheta_6\,\vartheta_4-\theta_5\,\theta_7\,\vartheta_0\,\vartheta_2-\theta_{13}\theta_{15}\vartheta_8\,\vartheta_{10}+\theta_9\,\theta_{11}\vartheta_{12}\vartheta_{14}$$

$$c_0c_3\,\Theta_7\,\Theta'_4=\theta_0\theta_3\,\vartheta_7\,\vartheta_4+\theta_7\,\theta_4\,\vartheta_0\,\vartheta_3+\theta_{13}\theta_{14}\vartheta_9\,\vartheta_{10}+\theta_9\,\theta_{10}\vartheta_{13}\vartheta_{14}$$

$$c_0c_{12}\Theta_8\,\Theta'_4=\theta_0\theta_{12}\vartheta_8\,\vartheta_4-\theta_2\,\theta_{14}\vartheta_6\,\vartheta_{10}+\theta_5\,\theta_9\,\vartheta_1\,\vartheta_{13}-\theta_7\,\theta_{11}\vartheta_3\,\vartheta_{15}$$

$$c_1c_{12}\Theta_9\,\Theta'_4=\theta_1\theta_{12}\vartheta_9\,\vartheta_4+\theta_8\,\theta_5\,\vartheta_0\,\vartheta_{13}-\theta_3\,\theta_{14}\vartheta_6\,\vartheta_{11}-\theta_7\,\theta_{10}\vartheta_2\,\vartheta_{15}$$

$$c_6c_8\,\Theta_{10}\Theta'_4=\theta_6\theta_8\,\vartheta_{10}\vartheta_4+\theta_{10}\theta_4\,\vartheta_6\,\vartheta_8+\theta_5\,\theta_{11}\vartheta_7\,\vartheta_9+\theta_7\,\theta_9\,\vartheta_5\,\vartheta_{11}$$

$$c_6c_9\,\Theta_{11}\Theta'_4=\theta_6\theta_9\,\vartheta_{11}\vartheta_4+\theta_4\,\theta_{11}\vartheta_6\,\vartheta_9+\theta_5\,\theta_{10}\vartheta_7\,\vartheta_8+\theta_7\,\theta_8\,\vartheta_5\,\vartheta_{10}$$

$$c_0c_8\,\Theta_{12}\Theta'_4=\theta_0\theta_8\,\vartheta_{12}\vartheta_4-\theta_{13}\theta_5\,\vartheta_1\,\vartheta_9-\theta_2\,\theta_{10}\vartheta_6\,\vartheta_{14}+\theta_7\,\theta_{15}\vartheta_{11}\vartheta_3$$

$$c_0c_9\,\Theta_{13}\Theta'_4=\theta_0\theta_9\,\vartheta_{13}\vartheta_4+\theta_4\,\theta_{13}\vartheta_0\,\vartheta_9-\theta_7\,\theta_{14}\vartheta_3\,\vartheta_{10}-\theta_3\,\theta_{10}\vartheta_7\,\vartheta_{14}$$

$$c_3c_9\,\Theta_{14}\Theta'_4=\theta_3\theta_9\,\vartheta_{14}\vartheta_4+\theta_4\,\theta_{14}\vartheta_3\,\vartheta_9-\theta_7\,\theta_{13}\vartheta_0\,\vartheta_{10}-\theta_0\,\theta_{10}\vartheta_7\,\vartheta_{13}$$

$$c_2c_9\,\Theta_{15}\Theta'_4=\theta_2\theta_9\,\vartheta_{15}\vartheta_4+\theta_7\,\theta_{12}\vartheta_1\,\vartheta_{10}+\theta_{14}\theta_5\,\vartheta_3\,\vartheta_8+\theta_0\,\theta_{11}\vartheta_6\,\vartheta_{13}$$

45. Sixth Set, with Θ'_5.

$$c_1c_4\,\Theta_0\,\Theta'_5=\theta_1\theta_4\,\vartheta_0\,\vartheta_5-\theta_0\,\theta_5\vartheta_1\vartheta_4-\theta_{10}\theta_{15}\vartheta_{11}\vartheta_{14}+\theta_{11}\theta_{14}\vartheta_{10}\vartheta_{15}$$
$$c_2c_6\,\Theta_1\,\Theta'_5=\theta_2\theta_6\,\vartheta_1\,\vartheta_5-\theta_1\,\theta_5\vartheta_2\vartheta_6-\theta_{10}\theta_{14}\vartheta_9\,\vartheta_{13}+\theta_9\,\theta_{13}\vartheta_{10}\vartheta_{14}$$
$$c_1c_6\,\Theta_2\,\Theta'_5=\theta_1\theta_6\,\vartheta_2\,\vartheta_5-\theta_2\,\theta_5\vartheta_1\vartheta_6-\theta_{10}\theta_{13}\vartheta_9\,\vartheta_{14}+\theta_9\,\theta_{14}\vartheta_{10}\vartheta_{13}$$
$$c_0c_6\,\Theta_3\,\Theta'_5=\theta_0\theta_6\,\vartheta_3\,\vartheta_5-\theta_3\,\theta_5\vartheta_0\vartheta_6-\theta_{11}\theta_{13}\vartheta_8\,\vartheta_{14}+\theta_8\,\theta_{14}\vartheta_{11}\vartheta_{13}$$
$$c_0c_1\,\Theta_4\,\Theta'_5=\theta_0\theta_1\,\vartheta_4\,\vartheta_5-\theta_4\,\theta_5\vartheta_0\vartheta_1-\theta_{14}\theta_{15}\vartheta_{10}\vartheta_{11}+\theta_{10}\theta_{11}\vartheta_{14}\vartheta_{15}$$
$$c_0c_0\,\Theta_5\,\Theta'_5=\theta_0\theta_0\,\vartheta_5\,\vartheta_5-\theta_5\,\theta_5\vartheta_0\vartheta_0-\theta_{11}\theta_{11}\vartheta_{14}\vartheta_{14}+\theta_{14}\theta_{14}\vartheta_{11}\vartheta_{11}$$
$$c_1c_2\,\Theta_6\,\Theta'_5=\theta_1\theta_2\,\vartheta_6\,\vartheta_5-\theta_6\,\theta_5\vartheta_1\vartheta_2-\theta_{13}\theta_{14}\vartheta_9\,\vartheta_{10}+\theta_9\,\theta_{10}\vartheta_{13}\vartheta_{14}$$
$$c_0c_2\,\Theta_7\,\Theta'_5=\theta_0\theta_2\,\vartheta_7\,\vartheta_5-\theta_7\,\theta_5\vartheta_0\vartheta_2-\theta_{13}\theta_{15}\vartheta_8\,\vartheta_{10}+\theta_8\,\theta_{10}\vartheta_{13}\vartheta_{15}$$
$$c_1c_{12}\Theta_8\,\Theta'_5=\theta_1\theta_{12}\vartheta_8\,\vartheta_5-\theta_8\,\theta_5\vartheta_1\vartheta_{12}-\theta_3\,\theta_{14}\vartheta_7\,\vartheta_{10}+\theta_7\,\theta_{10}\vartheta_3\,\vartheta_{14}$$
$$c_0c_{12}\Theta_9\,\Theta'_5=\theta_0\theta_{12}\vartheta_9\,\vartheta_5-\theta_9\,\theta_5\vartheta_0\vartheta_{12}-\theta_2\,\theta_{14}\vartheta_7\,\vartheta_{11}+\theta_7\,\theta_{11}\vartheta_2\,\vartheta_{14}$$
$$c_6c_9\,\Theta_{10}\Theta'_5=\theta_6\theta_9\,\vartheta_{10}\vartheta_5-\theta_{10}\theta_5\vartheta_6\vartheta_9-\theta_2\,\theta_{13}\vartheta_1\,\vartheta_{14}+\theta_1\,\theta_{14}\vartheta_2\,\vartheta_{13}$$
$$c_6c_8\,\Theta_{11}\Theta'_5=\theta_6\theta_8\,\vartheta_{11}\vartheta_5-\theta_{11}\theta_5\vartheta_6\vartheta_8-\theta_7\,\theta_9\,\vartheta_4\,\vartheta_{10}+\theta_4\,\theta_{10}\vartheta_7\,\vartheta_9$$
$$c_1c_8\,\Theta_{12}\Theta'_5=\theta_1\theta_8\,\vartheta_{12}\vartheta_5-\theta_{12}\theta_5\vartheta_1\vartheta_8-\theta_3\,\theta_{10}\vartheta_7\,\vartheta_{14}+\theta_7\,\theta_{14}\vartheta_3\,\vartheta_{10}$$
$$c_0c_8\,\Theta_{13}\Theta'_5=\theta_0\theta_8\,\vartheta_{13}\vartheta_5-\theta_{13}\theta_5\vartheta_0\vartheta_8-\theta_6\,\theta_{14}\vartheta_3\,\vartheta_{11}+\theta_3\,\theta_{11}\vartheta_6\,\vartheta_{14}$$
$$c_3c_8\,\Theta_{14}\Theta'_5=\theta_3\theta_8\,\vartheta_{14}\vartheta_5-\theta_{14}\theta_5\vartheta_3\vartheta_8-\theta_7\,\theta_{12}\vartheta_1\,\vartheta_{10}+\theta_1\,\theta_{10}\vartheta_7\,\vartheta_{12}$$
$$c_2c_8\,\Theta_{15}\Theta'_5=\theta_2\theta_8\,\vartheta_{15}\vartheta_5-\theta_{15}\theta_5\vartheta_2\vartheta_8-\theta_7\,\theta_{13}\vartheta_0\,\vartheta_{10}+\theta_0\,\theta_{10}\vartheta_7\,\vartheta_{13}$$

46. Seventh Set, with Θ'_6.

$$c_9c_{15}\Theta_0\,\Theta'_6=\theta_9\theta_{15}\vartheta_0\,\vartheta_6-\theta_1\,\theta_7\,\vartheta_8\vartheta_{14}-\theta_{13}\theta_{11}\vartheta_4\,\vartheta_2+\theta_3\,\theta_5\,\vartheta_{10}\vartheta_{12}$$
$$c_4c_3\,\Theta_1\,\Theta'_6=\theta_4\theta_3\,\vartheta_1\,\vartheta_6+\theta_2\,\theta_5\,\vartheta_0\vartheta_7-\theta_{11}\theta_{12}\vartheta_9\,\vartheta_{14}-\theta_{13}\theta_{10}\vartheta_8\,\vartheta_{15}$$
$$c_0c_4\,\Theta_2\,\Theta'_6=\theta_0\theta_4\,\vartheta_2\,\vartheta_6+\theta_1\,\theta_5\,\vartheta_3\vartheta_7+\theta_{10}\theta_{14}\vartheta_8\,\vartheta_{12}+\theta_{11}\theta_{15}\vartheta_9\,\vartheta_{13}$$
$$c_1c_4\,\Theta_3\,\Theta'_6=\theta_1\theta_4\,\vartheta_3\,\vartheta_6+\theta_0\,\theta_5\,\vartheta_2\vartheta_7+\theta_{11}\theta_{14}\vartheta_9\,\vartheta_{12}+\theta_{10}\theta_{15}\vartheta_8\,\vartheta_{13}$$
$$c_1c_3\,\Theta_4\,\Theta'_6=\theta_1\theta_3\,\vartheta_4\,\vartheta_6-\theta_5\,\theta_7\,\vartheta_0\vartheta_2+\theta_{13}\theta_{15}\vartheta_8\,\vartheta_{10}-\theta_9\,\theta_{11}\vartheta_{12}\vartheta_{14}$$
$$c_1c_2\,\Theta_5\,\Theta'_6=\theta_1\theta_2\,\vartheta_5\,\vartheta_6+\theta_5\,\theta_6\,\vartheta_1\vartheta_2-\theta_9\theta_{10}\vartheta_{13}\,\vartheta_{14}-\theta_{13}\theta_{14}\vartheta_9\,\vartheta_{10}$$
$$c_6c_6\,\Theta_6\,\Theta'_6=\theta_6\theta_6\,\vartheta_6\,\vartheta_6-\theta_5\,\theta_5\,\vartheta_5\vartheta_5+\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}-\theta_{14}\theta_{14}\vartheta_{14}\vartheta_{14}$$
$$c_0c_1\,\Theta_7\,\Theta'_6=\theta_0\theta_1\,\vartheta_7\,\vartheta_6+\theta_7\,\theta_6\,\vartheta_0\vartheta_1+\theta_{10}\theta_{11}\vartheta_{12}\vartheta_{13}+\theta_{12}\theta_{13}\vartheta_{10}\vartheta_{11}$$
$$c_2c_{12}\Theta_8\,\Theta'_6=\theta_2\theta_{12}\vartheta_8\,\vartheta_6+\theta_7\,\theta_9\,\vartheta_3\vartheta_{13}-\theta_0\,\theta_{14}\vartheta_4\,\vartheta_{10}-\theta_5\,\theta_{11}\vartheta_1\,\vartheta_{15}$$
$$c_0c_{15}\Theta_9\,\Theta'_6=\theta_0\theta_{15}\vartheta_9\,\vartheta_6-\theta_5\,\theta_{10}\vartheta_3\vartheta_{12}+\theta_2\,\theta_{13}\vartheta_4\,\vartheta_{11}-\theta_8\,\theta_7\,\vartheta_1\,\vartheta_{14}$$
$$c_0c_{12}\Theta_{10}\Theta'_6=\theta_0\theta_{12}\vartheta_{10}\vartheta_6+\theta_{10}\theta_6\,\vartheta_0\vartheta_{12}+\theta_1\,\theta_{13}\vartheta_7\,\vartheta_{11}+\theta_7\,\theta_{11}\vartheta_1\,\vartheta_{12}$$
$$c_9c_4\,\Theta_{11}\Theta'_6=\theta_9\theta_4\,\vartheta_{11}\vartheta_6+\theta_{11}\theta_6\,\vartheta_9\vartheta_4+\theta_7\,\theta_{10}\vartheta_5\,\vartheta_8+\theta_5\,\theta_8\,\vartheta_7\,\vartheta_{10}$$
$$c_2c_8\,\Theta_{12}\Theta'_6=\theta_2\theta_8\,\vartheta_{12}\vartheta_6-\theta_{13}\theta_7\,\vartheta_3\vartheta_9-\theta_0\,\theta_{10}\vartheta_4\,\vartheta_{14}+\theta_5\,\theta_{15}\vartheta_1\,\vartheta_{11}$$
$$c_3c_8\,\Theta_{13}\Theta'_6=\theta_3\theta_8\,\vartheta_{13}\vartheta_6+\theta_{13}\theta_6\,\vartheta_3\vartheta_8-\theta_{14}\theta_5\,\vartheta_0\,\vartheta_{11}-\theta_0\,\theta_{11}\vartheta_5\,\vartheta_{14}$$
$$c_0c_8\,\Theta_{14}\Theta'_6=\theta_0\theta_8\,\vartheta_{14}\vartheta_6+\theta_{14}\theta_6\,\vartheta_0\vartheta_8-\theta_5\,\theta_{13}\vartheta_3\,\vartheta_{11}-\theta_3\,\theta_{11}\vartheta_5\,\vartheta_{13}$$
$$c_0c_9\,\Theta_{15}\Theta'_6=\theta_0\theta_9\,\vartheta_{15}\vartheta_6+\theta_7\,\theta_{14}\vartheta_1\vartheta_8+\theta_4\,\theta_{13}\vartheta_2\,\vartheta_{11}+\theta_3\,\theta_{10}\vartheta_5\,\vartheta_{12}$$

47. Eighth Set, with Θ'_7.

$$c_1c_6\,\Theta_0\,\Theta'_7=\theta_1\theta_6\,\vartheta_0\vartheta_7-\theta_0\,\theta_7\vartheta_1\vartheta_6-\theta_{11}\theta_{12}\vartheta_{10}\vartheta_{13}+\theta_{10}\theta_{13}\vartheta_{11}\vartheta_{12}$$
$$c_0c_6\,\Theta_1\,\Theta'_7=\theta_0\theta_6\,\vartheta_1\vartheta_7-\theta_1\,\theta_7\vartheta_0\vartheta_6-\theta_{10}\theta_{12}\vartheta_{11}\vartheta_{13}+\theta_{11}\theta_{13}\vartheta_{10}\vartheta_{12}$$
$$c_1c_4\,\Theta_2\,\Theta'_7=\theta_1\theta_4\,\vartheta_2\vartheta_7-\theta_2\,\theta_7\vartheta_1\vartheta_4-\theta_8\,\theta_{13}\vartheta_{11}\vartheta_{14}+\theta_{11}\theta_{14}\vartheta_8\,\vartheta_{13}$$
$$c_0c_4\,\Theta_3\,\Theta'_7=\theta_0\theta_4\,\vartheta_3\vartheta_7-\theta_3\,\theta_7\vartheta_0\vartheta_4-\theta_9\,\theta_{13}\vartheta_{10}\vartheta_{14}+\theta_{10}\theta_{14}\vartheta_9\,\vartheta_{13}$$
$$c_0c_3\,\Theta_4\,\Theta'_7=\theta_0\theta_3\,\vartheta_4\vartheta_7-\theta_4\,\theta_7\vartheta_0\vartheta_3-\theta_9\,\theta_{10}\vartheta_{13}\vartheta_{14}+\theta_{13}\theta_{14}\vartheta_9\,\vartheta_{10}$$
$$c_0c_2\,\Theta_5\,\Theta'_7=\theta_0\theta_2\,\vartheta_5\vartheta_7-\theta_5\,\theta_7\vartheta_0\vartheta_2-\theta_8\,\theta_{10}\vartheta_{13}\vartheta_{15}+\theta_{13}\theta_{15}\vartheta_8\,\vartheta_{10}$$
$$c_0c_1\,\Theta_6\,\Theta'_7=\theta_0\theta_1\,\vartheta_6\vartheta_7-\theta_6\,\theta_7\vartheta_0\vartheta_1-\theta_{12}\theta_{13}\vartheta_{10}\vartheta_{11}+\theta_{10}\theta_{11}\vartheta_{12}\vartheta_{13}$$
$$c_2c_2\,\Theta_7\,\Theta'_7=\theta_2\theta_2\,\vartheta_7\vartheta_7-\theta_7\,\theta_7\vartheta_2\vartheta_2-\theta_{14}\theta_{14}\vartheta_{11}\vartheta_{11}+\theta_{11}\theta_{11}\vartheta_{14}\vartheta_{14}$$
$$c_3c_{12}\Theta_8\,\Theta'_7=\theta_3\theta_{12}\vartheta_8\,\vartheta_7-\theta_8\,\theta_7\vartheta_3\vartheta_{12}-\theta_1\,\theta_{14}\vartheta_{10}\vartheta_5+\theta_5\,\theta_{10}\vartheta_1\,\vartheta_{14}$$
$$c_2c_{12}\Theta_9\,\Theta'_7=\theta_2\theta_{12}\vartheta_9\,\vartheta_7-\theta_9\,\theta_7\vartheta_2\vartheta_{12}-\theta_0\,\theta_{14}\vartheta_{11}\vartheta_5+\theta_5\,\theta_{11}\vartheta_0\,\vartheta_{14}$$
$$c_4c_9\,\Theta_{10}\Theta'_7=\theta_4\theta_9\,\vartheta_{10}\vartheta_7-\theta_{10}\theta_7\vartheta_4\vartheta_9-\theta_5\,\theta_8\,\vartheta_6\,\vartheta_{11}+\theta_6\,\theta_{11}\vartheta_5\,\vartheta_8$$
$$c_0c_{12}\Theta_{11}\Theta'_7=\theta_0\theta_{12}\vartheta_{11}\vartheta_7-\theta_{11}\theta_7\vartheta_0\vartheta_{12}-\theta_1\,\theta_{13}\vartheta_6\,\vartheta_{10}+\theta_6\,\theta_{10}\vartheta_1\,\vartheta_{13}$$
$$c_2c_9\,\Theta_{12}\Theta'_7=\theta_2\theta_9\,\vartheta_{12}\vartheta_7-\theta_{12}\theta_7\vartheta_2\vartheta_9-\theta_0\,\theta_{11}\vartheta_5\,\vartheta_{14}+\theta_5\,\theta_{14}\vartheta_0\,\vartheta_{11}$$
$$c_2c_8\,\Theta_{13}\Theta'_7=\theta_2\theta_8\,\vartheta_{13}\vartheta_7-\theta_{13}\theta_7\vartheta_2\vartheta_8-\theta_5\,\theta_{15}\vartheta_0\,\vartheta_{10}+\theta_0\,\theta_{10}\vartheta_5\,\vartheta_{15}$$
$$c_0c_9\,\Theta_{14}\Theta'_7=\theta_0\theta_9\,\vartheta_{14}\vartheta_7-\theta_{14}\theta_7\vartheta_0\vartheta_9-\theta_5\,\theta_{12}\vartheta_2\,\vartheta_{11}+\theta_2\,\theta_{11}\vartheta_5\,\vartheta_{12}$$
$$c_0c_8\,\Theta_{15}\Theta'_7=\theta_0\theta_8\,\vartheta_{15}\vartheta_7-\theta_{15}\theta_7\vartheta_0\vartheta_8-\theta_5\,\theta_{13}\vartheta_2\,\vartheta_{10}+\theta_2\,\theta_{10}\vartheta_5\,\vartheta_{13}$$

48. Ninth Set, with Θ'_8.

$$c_4c_{12}\Theta_0\,\Theta'_8=\theta_4\theta_{12}\vartheta_0\,\vartheta_8+\theta_5\,\theta_{13}\vartheta_9\vartheta_1+\theta_6\,\theta_{14}\vartheta_2\vartheta_{10}+\theta_7\,\theta_{15}\vartheta_3\,\vartheta_{11}$$
$$c_0c_9\,\Theta_1\,\Theta'_8=\theta_0\theta_9\,\vartheta_1\,\vartheta_8+\theta_3\,\theta_{10}\vartheta_2\vartheta_{11}+\theta_4\,\theta_{13}\vartheta_5\vartheta_{12}+\theta_7\,\theta_{14}\vartheta_6\,\vartheta_{15}$$
$$c_3c_9\,\Theta_2\,\Theta'_8=\theta_3\theta_9\,\vartheta_2\,\vartheta_8+\theta_{11}\theta_1\,\vartheta_0\vartheta_{10}-\theta_7\,\theta_{13}\vartheta_6\vartheta_{12}-\theta_5\,\theta_{15}\vartheta_4\,\vartheta_{14}$$
$$c_2c_9\,\Theta_3\,\Theta'_8=\theta_2\theta_9\,\vartheta_3\,\vartheta_8+\theta_0\,\theta_{11}\vartheta_1\vartheta_{10}+\theta_7\,\theta_{12}\vartheta_6\vartheta_{13}+\theta_5\,\theta_{14}\vartheta_4\,\vartheta_{15}$$
$$c_0c_{12}\Theta_4\,\Theta'_8=\theta_0\theta_{12}\vartheta_4\,\vartheta_8+\theta_2\,\theta_{14}\vartheta_6\vartheta_{10}-\theta_5\,\theta_9\,\vartheta_1\vartheta_{13}-\theta_7\,\theta_{11}\vartheta_3\,\vartheta_{15}$$
$$c_1c_{12}\Theta_5\,\Theta'_8=\theta_1\theta_{12}\vartheta_5\,\vartheta_8+\theta_5\,\theta_8\,\vartheta_1\vartheta_{12}+\theta_3\,\theta_{14}\vartheta_7\vartheta_{10}+\theta_7\,\theta_{10}\vartheta_3\,\vartheta_{14}$$
$$c_2c_{12}\Theta_6\,\Theta'_8=\theta_2\theta_{12}\vartheta_6\,\vartheta_8-\theta_7\,\theta_9\,\vartheta_3\vartheta_{13}+\theta_0\,\theta_{14}\vartheta_4\vartheta_{10}-\theta_5\,\theta_{11}\vartheta_1\,\vartheta_{15}$$
$$c_3c_{12}\Theta_7\,\Theta'_8=\theta_3\theta_{12}\vartheta_7\,\vartheta_8+\theta_7\,\theta_8\,\vartheta_3\vartheta_{12}+\theta_1\,\theta_{14}\vartheta_5\vartheta_{10}+\theta_5\,\theta_{10}\vartheta_1\,\vartheta_{14}$$
$$c_8c_8\,\Theta_8\,\Theta'_8=\theta_8\theta_8\,\vartheta_8\,\vartheta_8+\theta_5\,\theta_5\,\vartheta_5\vartheta_5-\theta_7\,\theta_7\,\vartheta_7\vartheta_7-\theta_{10}\theta_{10}\vartheta_{10}\vartheta_{10}$$
$$c_0c_1\,\Theta_9\,\Theta'_8=\theta_0\theta_1\,\vartheta_9\,\vartheta_8-\theta_{10}\theta_{11}\vartheta_2\vartheta_3-\theta_{14}\theta_{15}\vartheta_6\vartheta_7+\theta_4\,\theta_5\,\vartheta_{12}\vartheta_{13}$$
$$c_0c_2\,\Theta_{10}\Theta'_8=\theta_0\theta_2\,\vartheta_{10}\vartheta_8+\theta_{10}\theta_8\,\vartheta_0\vartheta_2+\theta_{13}\theta_{15}\vartheta_5\vartheta_7+\theta_5\,\theta_7\,\vartheta_{13}\vartheta_{15}$$
$$c_0c_3\,\Theta_{11}\Theta'_8=\theta_0\theta_3\,\vartheta_{11}\vartheta_8+\theta_{11}\theta_8\,\vartheta_0\vartheta_3+\theta_{13}\theta_{14}\vartheta_5\vartheta_6+\theta_5\,\theta_6\,\vartheta_{13}\vartheta_{14}$$
$$c_0c_4\,\Theta_{12}\Theta'_8=\theta_0\theta_4\,\vartheta_{12}\vartheta_8-\theta_{10}\theta_{14}\vartheta_2\vartheta_6+\theta_{11}\theta_{15}\vartheta_3\vartheta_7-\theta_1\,\theta_5\,\vartheta_9\,\vartheta_{13}$$
$$c_1c_4\,\Theta_{13}\Theta'_8=\theta_1\theta_4\,\vartheta_{13}\vartheta_8+\theta_{13}\theta_8\,\vartheta_1\vartheta_4-\theta_{11}\theta_{14}\vartheta_7\vartheta_2-\theta_2\theta_7\,\vartheta_{11}\vartheta_{14}$$
$$c_0c_6\,\Theta_{14}\Theta'_8=\theta_0\theta_6\,\vartheta_{14}\vartheta_8+\theta_{14}\theta_8\,\vartheta_0\vartheta_6-\theta_{11}\theta_{13}\vartheta_3\vartheta_5-\theta_3\theta_5\,\vartheta_{11}\vartheta_{13}$$
$$c_3c_4\,\Theta_{15}\Theta'_8=\theta_3\theta_4\,\vartheta_{15}\vartheta_8+\theta_{10}\theta_{13}\vartheta_1\vartheta_6+\theta_9\,\theta_{14}\vartheta_2\vartheta_5+\theta_0\theta_7\,\vartheta_{11}\vartheta_{12}$$

49. Tenth Set—with Θ'_9.

$$c_6c_{15}\Theta_0\,\Theta'_9=\theta_6\theta_{15}\vartheta_0\vartheta_9-\theta_{14}\theta_7\vartheta_8\vartheta_1-\theta_2\theta_{11}\vartheta_4\vartheta_{13}+\theta_3\theta_{10}\vartheta_{12}\vartheta_5$$
$$c_0c_8\Theta_1\,\Theta'_9=\theta_0\theta_8\vartheta_1\vartheta_9+\theta_2\theta_{10}\vartheta_3\vartheta_{11}+\theta_5\theta_{13}\vartheta_4\vartheta_{12}+\theta_7\theta_{15}\vartheta_6\vartheta_{14}$$
$$c_3c_8\Theta_2\,\Theta'_9=\theta_3\theta_8\vartheta_2\vartheta_9+\theta_0\theta_{11}\vartheta_1\vartheta_{10}-\theta_6\theta_{13}\vartheta_7\vartheta_{12}-\theta_5\theta_{14}\vartheta_4\vartheta_{15}$$
$$c_2c_8\Theta_3\,\Theta'_9=\theta_2\theta_8\vartheta_3\vartheta_9+\theta_0\theta_{10}\vartheta_1\vartheta_{11}+\theta_7\theta_{13}\vartheta_6\vartheta_{12}+\theta_4\theta_{14}\vartheta_5\vartheta_{15}$$
$$c_1c_{12}\Theta_4\,\Theta'_9=\theta_1\theta_{12}\vartheta_4\vartheta_9-\theta_8\theta_5\vartheta_0\vartheta_{13}+\theta_3\theta_{14}\vartheta_6\vartheta_{11}-\theta_7\theta_{10}\vartheta_2\vartheta_{15}$$
$$c_0c_{12}\Theta_5\,\Theta'_9=\theta_0\theta_{12}\vartheta_5\vartheta_9+\theta_5\theta_9\vartheta_0\vartheta_{12}+\theta_2\theta_{14}\vartheta_7\vartheta_{11}+\theta_7\theta_{11}\vartheta_2\vartheta_{14}$$
$$c_0c_{15}\Theta_6\,\Theta'_9=\theta_0\theta_{15}\vartheta_6\vartheta_9-\theta_5\theta_{10}\vartheta_3\vartheta_{12}-\theta_2\theta_{13}\vartheta_4\vartheta_{11}+\theta_7\theta_8\vartheta_1\vartheta_{14}$$
$$c_2c_{12}\Theta_7\,\Theta'_9=\theta_2\theta_{12}\vartheta_7\vartheta_9+\theta_7\theta_9\vartheta_2\vartheta_{12}+\theta_0\theta_{14}\vartheta_5\vartheta_{11}+\theta_5\theta_{11}\vartheta_0\vartheta_{14}$$
$$c_0c_1\Theta_8\,\Theta'_9=\theta_0\theta_1\vartheta_8\vartheta_9-\theta_{10}\theta_{11}\vartheta_2\vartheta_3+\theta_{14}\theta_{15}\vartheta_6\vartheta_7-\theta_4\theta_5\vartheta_{12}\vartheta_{13}$$
$$c_9c_9\Theta_9\,\Theta'_9=\theta_9\theta_9\vartheta_9\vartheta_9-\theta_{10}\theta_{10}\vartheta_{10}\vartheta_{10}-\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}+\theta_{14}\theta_{14}\vartheta_{14}\vartheta_{14}$$
$$c_0c_3\Theta_{10}\Theta'_9=\theta_0\theta_3\vartheta_{10}\vartheta_9+\theta_{10}\theta_9\vartheta_0\vartheta_3-\theta_{13}\theta_{14}\vartheta_4\vartheta_7-\theta_4\theta_7\vartheta_{13}\vartheta_{14}$$
$$c_0c_2\Theta_{11}\Theta'_9=\theta_0\theta_2\vartheta_{11}\vartheta_9+\theta_{11}\theta_9\vartheta_0\vartheta_2+\theta_{12}\theta_{14}\vartheta_5\vartheta_7+\theta_5\theta_7\vartheta_{12}\vartheta_{14}$$
$$c_9c_{12}\Theta_{12}\Theta'_9=\theta_9\theta_{12}\vartheta_{12}\vartheta_9+\theta_0\theta_5\vartheta_0\vartheta_5-\theta_2\theta_7\vartheta_2\vartheta_7-\theta_{11}\theta_{14}\vartheta_{11}\vartheta_{14}$$
$$c_0c_4\Theta_{13}\Theta'_9=\theta_0\theta_4\vartheta_{13}\vartheta_9+\theta_{13}\theta_9\vartheta_0\vartheta_4-\theta_{11}\theta_{15}\vartheta_6\vartheta_2-\theta_2\theta_6\vartheta_{11}\vartheta_{15}$$
$$c_3c_4\Theta_{14}\Theta'_9=\theta_3\theta_4\vartheta_{14}\vartheta_9+\theta_{14}\theta_9\vartheta_3\vartheta_4-\theta_{10}\theta_{13}\vartheta_0\vartheta_7-\theta_0\theta_7\vartheta_{10}\vartheta_{13}$$
$$c_0c_6\Theta_{15}\Theta'_9=\theta_0\theta_6\vartheta_{15}\vartheta_9+\theta_{11}\theta_{13}\vartheta_2\vartheta_4+\theta_8\theta_{14}\vartheta_1\vartheta_7+\theta_3\theta_5\vartheta_{10}\vartheta_{12}$$

50. Eleventh Set—with Θ'_{10}.

$$c_2c_8\Theta_0\,\Theta'_{10}=\theta_2\theta_8\vartheta_0\vartheta_{10}-\theta_0\theta_{10}\vartheta_2\vartheta_8-\theta_5\theta_{15}\vartheta_7\vartheta_{13}+\theta_7\theta_{13}\vartheta_5\vartheta_{15}$$
$$c_3c_8\Theta_1\,\Theta'_{10}=\theta_3\theta_8\vartheta_1\vartheta_{10}-\theta_1\theta_{10}\vartheta_3\vartheta_8-\theta_5\theta_{14}\vartheta_7\vartheta_{12}+\theta_7\theta_{12}\vartheta_5\vartheta_{14}$$
$$c_0c_8\Theta_2\,\Theta'_{10}=\theta_0\theta_8\vartheta_2\vartheta_{10}-\theta_2\theta_{10}\vartheta_0\vartheta_8-\theta_7\theta_{15}\vartheta_5\vartheta_{13}+\theta_5\theta_{13}\vartheta_7\vartheta_{15}$$
$$c_1c_8\Theta_3\,\Theta'_{10}=\theta_1\theta_8\vartheta_3\vartheta_{10}-\theta_3\theta_{10}\vartheta_1\vartheta_8-\theta_7\theta_{14}\vartheta_5\vartheta_{12}+\theta_5\theta_{12}\vartheta_7\vartheta_{14}$$
$$c_6c_8\Theta_4\,\Theta'_{10}=\theta_6\theta_8\vartheta_4\vartheta_{10}-\theta_4\theta_{10}\vartheta_6\vartheta_8-\theta_7\theta_9\vartheta_5\vartheta_{11}+\theta_5\theta_{11}\vartheta_7\vartheta_9$$
$$c_6c_9\Theta_5\,\Theta'_{10}=\theta_6\theta_9\vartheta_5\vartheta_{10}-\theta_5\theta_{10}\vartheta_6\vartheta_9-\theta_1\theta_{14}\vartheta_7\vartheta_{13}+\theta_2\theta_{13}\vartheta_1\vartheta_{14}$$
$$c_0c_{12}\Theta_6\,\Theta'_{10}=\theta_0\theta_{12}\vartheta_6\vartheta_{10}-\theta_6\theta_{10}\vartheta_0\vartheta_{12}-\theta_1\theta_{13}\vartheta_7\vartheta_{11}+\theta_7\theta_{11}\vartheta_1\vartheta_{13}$$
$$c_4c_9\Theta_7\,\Theta'_{10}=\theta_4\theta_9\vartheta_7\vartheta_{10}-\theta_7\theta_{10}\vartheta_4\vartheta_9-\theta_6\theta_{11}\vartheta_5\vartheta_8+\theta_5\theta_8\vartheta_6\vartheta_{11}$$
$$c_0c_2\Theta_8\,\Theta'_{10}=\theta_0\theta_2\vartheta_8\vartheta_{10}-\theta_8\theta_{10}\vartheta_0\vartheta_2-\theta_{13}\theta_{15}\vartheta_5\vartheta_7+\theta_5\theta_7\vartheta_{13}\vartheta_{15}$$
$$c_0c_3\Theta_9\,\Theta'_{10}=\theta_0\theta_3\vartheta_9\vartheta_{10}-\theta_9\theta_{10}\vartheta_0\vartheta_3-\theta_{13}\theta_{14}\vartheta_4\vartheta_7+\theta_4\theta_7\vartheta_{13}\vartheta_{14}$$
$$c_{15}c_{15}\Theta_{10}\Theta'_{10}=\theta_{15}\theta_{15}\vartheta_{10}\vartheta_{10}-\theta_{10}\theta_{10}\vartheta_{15}\vartheta_{15}-\theta_{11}\theta_{11}\vartheta_{14}\vartheta_{14}+\theta_{14}\theta_{14}\vartheta_{11}\vartheta_{11}$$
$$c_0c_1\Theta_{11}\Theta'_{10}=\theta_0\theta_1\vartheta_{11}\vartheta_{10}-\theta_{11}\theta_{10}\vartheta_0\vartheta_1-\theta_6\theta_7\vartheta_{12}\vartheta_{13}+\theta_{12}\theta_{13}\vartheta_6\vartheta_7$$
$$c_9c_{15}\Theta_{12}\Theta'_{10}=\theta_9\theta_{15}\vartheta_{12}\vartheta_{10}-\theta_{12}\theta_{10}\vartheta_9\vartheta_{15}-\theta_{11}\theta_{13}\vartheta_8\vartheta_{14}+\theta_8\theta_{14}\vartheta_{11}\vartheta_{13}$$
$$c_3c_4\Theta_{13}\Theta'_{10}=\theta_3\theta_4\vartheta_{13}\vartheta_{10}-\theta_{13}\theta_{10}\vartheta_3\vartheta_4-\theta_{11}\theta_{12}\vartheta_2\vartheta_5+\theta_2\theta_5\vartheta_{11}\vartheta_{12}$$
$$c_0c_4\Theta_{14}\Theta'_{10}=\theta_0\theta_4\vartheta_{14}\vartheta_{10}-\theta_{14}\theta_{10}\vartheta_0\vartheta_4-\theta_9\theta_{13}\vartheta_3\vartheta_7+\theta_3\theta_7\vartheta_9\vartheta_{13}$$
$$c_1c_4\Theta_{15}\Theta'_{10}=\theta_1\theta_4\vartheta_{15}\vartheta_{10}-\theta_{15}\theta_{10}\vartheta_1\vartheta_4-\theta_{11}\theta_{14}\vartheta_0\vartheta_5+\theta_0\theta_5\vartheta_{11}\vartheta_{14}$$

51. Twelfth Set, with Θ'_{11}.

$$c_3c_8\,\Theta_0\,\Theta'_{11}=\theta_3\theta_8\,\vartheta_0\,\vartheta_{11}-\theta_0\,\theta_{11}\vartheta_3\vartheta_8-\theta_6\,\theta_{13}\vartheta_5\,\vartheta_{14}+\theta_5\,\theta_{14}\vartheta_6\,\vartheta_{13}$$
$$c_3c_9\,\Theta_1\,\Theta'_{11}=\theta_3\theta_9\,\vartheta_1\,\vartheta_{11}-\theta_1\,\theta_{11}\vartheta_3\vartheta_9-\theta_7\,\theta_{13}\vartheta_5\,\vartheta_{15}+\theta_5\,\theta_{15}\vartheta_7\,\vartheta_{13}$$
$$c_0c_9\,\Theta_2\,\Theta'_{11}=\theta_0\theta_9\,\vartheta_2\,\vartheta_{11}-\theta_2\,\theta_{11}\vartheta_0\vartheta_9-\theta_5\,\theta_{12}\vartheta_7\,\vartheta_{14}+\theta_7\,\theta_{14}\vartheta_5\,\vartheta_{12}$$
$$c_0c_8\,\Theta_3\,\Theta'_{11}=\theta_0\theta_8\,\vartheta_3\,\vartheta_{11}-\theta_3\,\theta_{11}\vartheta_0\vartheta_8-\theta_6\,\theta_{14}\vartheta_5\,\vartheta_{13}+\theta_5\,\theta_{13}\vartheta_6\,\vartheta_{14}$$
$$c_6c_9\,\Theta_4\,\Theta'_{11}=\theta_6\theta_9\,\vartheta_4\,\vartheta_{11}-\theta_4\,\theta_{11}\vartheta_6\vartheta_9-\theta_7\,\theta_8\,\vartheta_5\,\vartheta_{10}+\theta_5\,\theta_{10}\vartheta_7\,\vartheta_8$$
$$c_6c_8\,\Theta_5\,\Theta'_{11}=\theta_6\theta_8\,\vartheta_5\,\vartheta_{11}-\theta_5\,\theta_{11}\vartheta_6\vartheta_8-\theta_4\,\theta_{10}\vartheta_7\,\vartheta_9+\theta_7\,\theta_9\,\vartheta_4\,\vartheta_{10}$$
$$c_4c_9\,\Theta_6\,\Theta'_{11}=\theta_4\theta_9\,\vartheta_6\,\vartheta_{11}-\theta_6\,\theta_{11}\vartheta_4\vartheta_9-\theta_5\,\theta_8\,\vartheta_7\,\vartheta_{10}+\theta_7\,\theta_{10}\vartheta_5\,\vartheta_8$$
$$c_0c_{12}\Theta_7\,\Theta'_{11}=\theta_0\theta_{12}\vartheta_7\,\vartheta_{11}-\theta_7\,\theta_{11}\vartheta_0\vartheta_{12}-\theta_6\,\theta_{10}\vartheta_1\,\vartheta_{13}+\theta_1\,\theta_{13}\vartheta_6\,\vartheta_{10}$$
$$c_0c_3\,\Theta_8\,\Theta'_{11}=\theta_0\theta_3\,\vartheta_8\,\vartheta_{11}-\theta_8\,\theta_{11}\vartheta_0\vartheta_3-\theta_5\,\theta_6\,\vartheta_{13}\vartheta_{14}+\theta_{13}\theta_{14}\vartheta_5\,\vartheta_6$$
$$c_0c_2\,\Theta_9\,\Theta'_{11}=\theta_0\theta_2\,\vartheta_9\,\vartheta_{11}-\theta_9\,\theta_{11}\vartheta_0\vartheta_2-\theta_{12}\theta_{14}\vartheta_5\,\vartheta_7+\theta_5\,\theta_7\,\vartheta_{12}\vartheta_{14}$$
$$c_0c_1\,\Theta_{10}\Theta'_{11}=\theta_0\theta_1\,\vartheta_{10}\vartheta_{11}-\theta_{10}\theta_{11}\vartheta_0\vartheta_1-\theta_{12}\theta_{13}\vartheta_6\,\vartheta_7+\theta_6\,\theta_7\,\vartheta_{12}\vartheta_{13}$$
$$c_1c_1\,\Theta_{11}\Theta'_{11}=\theta_1\theta_1\,\vartheta_{11}\vartheta_{11}-\theta_{11}\theta_{11}\vartheta_1\vartheta_1-\theta_{13}\theta_{13}\vartheta_7\,\vartheta_7+\theta_7\,\theta_7\,\vartheta_{13}\vartheta_{13}$$
$$c_1c_6\,\Theta_{12}\Theta'_{11}=\theta_1\theta_6\,\vartheta_{12}\vartheta_{11}-\theta_{12}\theta_{11}\vartheta_1\vartheta_6-\theta_0\,\theta_2\,\vartheta_{10}\vartheta_{13}+\theta_{10}\theta_{13}\vartheta_0\,\vartheta_7$$
$$c_0c_6\,\Theta_{13}\Theta'_{11}=\theta_0\theta_6\,\vartheta_{13}\vartheta_{11}-\theta_{13}\theta_{11}\vartheta_0\vartheta_6-\theta_{10}\theta_{12}\vartheta_1\,\vartheta_7+\theta_1\,\theta_7\,\vartheta_{10}\vartheta_{12}$$
$$c_1c_4\,\Theta_{14}\Theta'_{11}=\theta_1\theta_4\,\vartheta_{14}\vartheta_{11}-\theta_{14}\theta_{11}\vartheta_1\vartheta_4-\theta_0\,\theta_5\,\vartheta_{10}\vartheta_{15}+\theta_{10}\theta_{15}\vartheta_0\,\vartheta_5$$
$$c_0c_4\,\Theta_{15}\Theta'_{11}=\theta_0\theta_4\,\vartheta_{15}\vartheta_{11}-\theta_{15}\theta_{11}\vartheta_0\vartheta_4-\theta_{10}\theta_{14}\vartheta_1\,\vartheta_5+\theta_1\,\theta_5\,\vartheta_{10}\vartheta_{14}$$

52. Thirteenth Set, with Θ'_{12}.

$$c_4\,c_8\,\Theta_0\,\Theta'_{12}=\theta_4\,\theta_8\,\vartheta_0\,\vartheta_{12}+\theta_1\,\theta_{13}\vartheta_5\,\vartheta_9-\theta_6\,\theta_{10}\vartheta_2\,\vartheta_{14}-\theta_7\,\theta_{11}\vartheta_3\,\vartheta_{15}$$
$$c_4\,c_9\,\Theta_1\,\Theta'_{12}=\theta_4\,\theta_9\,\vartheta_1\,\vartheta_{12}+\theta_0\,\theta_{13}\vartheta_5\,\vartheta_8+\theta_3\,\theta_{14}\vartheta_6\,\vartheta_{11}+\theta_7\,\theta_{10}\vartheta_2\,\vartheta_{15}$$
$$c_6\,c_8\,\Theta_2\,\Theta'_{12}=\theta_6\,\theta_8\,\vartheta_2\,\vartheta_{12}+\theta_3\,\theta_{13}\vartheta_7\,\vartheta_9+\theta_0\,\theta_{14}\vartheta_4\,\vartheta_{10}+\theta_5\,\theta_{11}\vartheta_1\,\vartheta_{15}$$
$$c_0\,c_{15}\Theta_3\,\Theta'_{12}=\theta_0\,\theta_{15}\vartheta_3\,\vartheta_{12}-\theta_5\,\theta_{10}\vartheta_6\,\vartheta_9+\theta_4\,\theta_{11}\vartheta_7\,\vartheta_8-\theta_6\,\theta_{14}\vartheta_2\,\vartheta_{13}$$
$$c_0\,c_8\,\Theta_4\,\Theta'_{12}=\theta_0\,\theta_8\,\vartheta_4\,\vartheta_{12}-\theta_{13}\theta_5\,\vartheta_1\,\vartheta_9-\theta_7\,\theta_{15}\vartheta_3\,\vartheta_{11}+\theta_2\,\theta_{10}\vartheta_6\,\vartheta_{14}$$
$$c_1\,c_8\,\Theta_5\,\Theta'_{12}=\theta_1\,\theta_8\,\vartheta_5\,\vartheta_{12}+\theta_5\,\theta_{12}\vartheta_1\,\vartheta_8+\theta_{14}\theta_7\,\vartheta_3\,\vartheta_{10}+\theta_3\,\theta_{10}\vartheta_7\,\vartheta_{14}$$
$$c_2\,c_8\,\Theta_6\,\Theta'_{12}=\theta_2\,\theta_8\,\vartheta_6\,\vartheta_{12}-\theta_{13}\theta_7\,\vartheta_3\,\vartheta_9-\theta_5\,\theta_{15}\vartheta_1\,\vartheta_{11}+\theta_0\,\theta_{10}\vartheta_4\,\vartheta_{14}$$
$$c_2\,c_9\,\Theta_7\,\Theta'_{12}=\theta_2\,\theta_9\,\vartheta_7\,\vartheta_{12}+\theta_7\,\theta_{12}\vartheta_2\,\vartheta_9+\theta_5\,\theta_{14}\vartheta_0\,\vartheta_{11}+\theta_0\,\theta_{11}\vartheta_5\,\vartheta_{14}$$
$$c_0\,c_4\,\Theta_8\,\Theta'_{12}=\theta_0\,\theta_4\,\vartheta_8\,\vartheta_{12}+\theta_1\,\theta_5\,\vartheta_9\,\vartheta_{13}-\theta_{10}\theta_{14}\vartheta_2\,\vartheta_6-\theta_{11}\theta_{15}\vartheta_3\,\vartheta_7$$
$$c_9\,c_{12}\Theta_9\,\Theta'_{12}=\theta_9\,\theta_{12}\vartheta_9\,\vartheta_{12}+\theta_0\,\theta_5\,\vartheta_0\,\vartheta_5-\theta_2\,\theta_7\,\vartheta_2\,\vartheta_7-\theta_{11}\theta_{14}\vartheta_{11}\vartheta_{14}$$
$$c_9\,c_{15}\Theta_{10}\Theta'_{12}=\theta_9\,\theta_{15}\vartheta_{10}\vartheta_{12}+\theta_{10}\theta_{12}\vartheta_9\,\vartheta_{15}-\theta_8\,\theta_{14}\vartheta_{11}\vartheta_{13}-\theta_{11}\theta_{13}\vartheta_8\,\vartheta_{14}$$
$$c_1\,c_6\,\Theta_{11}\Theta'_{12}=\theta_1\,\theta_6\,\vartheta_{11}\vartheta_{12}+\theta_{11}\theta_{12}\vartheta_1\,\vartheta_6+\theta_{10}\theta_{13}\vartheta_0\,\vartheta_7+\theta_0\,\theta_7\,\vartheta_{10}\vartheta_{13}$$
$$c_{12}c_{12}\Theta_{12}\Theta'_{12}=\theta_{12}\theta_{12}\vartheta_{12}\vartheta_{12}-\theta_{10}\theta_{10}\vartheta_{10}\vartheta_{10}+\theta_{11}\theta_{11}\vartheta_{11}\vartheta_{11}-\theta_{13}\theta_{13}\vartheta_{13}\vartheta_{13}$$
$$c_8\,c_9\,\Theta_{13}\Theta'_{12}=\theta_8\,\theta_9\,\vartheta_{13}\vartheta_{12}+\theta_{13}\theta_{12}\vartheta_8\,\vartheta_9-\theta_{14}\theta_{15}\vartheta_{10}\vartheta_{11}-\theta_{10}\theta_{11}\vartheta_{14}\vartheta_{15}$$
$$c_0\,c_2\,\Theta_{14}\Theta'_{12}=\theta_0\,\theta_2\,\vartheta_{14}\vartheta_{12}+\theta_{14}\theta_{12}\vartheta_0\,\vartheta_2-\theta_7\,\theta_5\,\vartheta_9\,\vartheta_{11}-\theta_9\,\theta_{11}\vartheta_5\,\vartheta_7$$
$$c_0\,c_3\,\Theta_{15}\Theta'_{12}=\theta_0\,\theta_3\,\vartheta_{15}\vartheta_{12}+\theta_{13}\theta_{14}\vartheta_1\,\vartheta_2+\theta_8\,\theta_{11}\vartheta_7\,\vartheta_4+\theta_5\,\theta_6\,\vartheta_9\,\vartheta_{10}$$

53. FOURTEENTH Set, with Θ'_{13}.

$$c_1c_{12}\Theta_0\,\Theta'_{13}=\theta_1\theta_{12}\vartheta_0\,\vartheta_{13}-\theta_0\,\theta_{13}\vartheta_1\vartheta_{12}-\theta_7\,\theta_{10}\vartheta_6\,\vartheta_{11}+\theta_6\,\theta_{11}\vartheta_7\,\vartheta_{10}$$
$$c_0c_{12}\Theta_1\,\Theta'_{13}=\theta_0\theta_{12}\vartheta_1\,\vartheta_{13}-\theta_1\,\theta_{13}\vartheta_0\vartheta_{12}-\theta_7\,\theta_{11}\vartheta_6\,\vartheta_{10}+\theta_6\,\theta_{10}\vartheta_7\,\vartheta_{11}$$
$$c_3c_{12}\Theta_2\,\Theta'_{13}=\theta_3\theta_{12}\vartheta_2\,\vartheta_{13}-\theta_2\,\theta_{13}\vartheta_3\vartheta_{12}-\theta_5\,\theta_{10}\vartheta_4\,\vartheta_{11}+\theta_4\,\theta_{11}\vartheta_5\,\vartheta_{10}$$
$$c_2c_{12}\Theta_3\,\Theta'_{13}=\theta_2\theta_{12}\vartheta_3\,\vartheta_{13}-\theta_3\,\theta_{13}\vartheta_2\vartheta_{12}-\theta_5\,\theta_{11}\vartheta_4\,\vartheta_{10}+\theta_4\,\theta_{10}\vartheta_5\,\vartheta_{11}$$
$$c_0c_9\,\Theta_4\,\Theta'_{13}=\theta_0\theta_9\,\vartheta_4\,\vartheta_{13}-\theta_4\,\theta_{13}\vartheta_0\,\vartheta_9-\theta_7\,\theta_{14}\vartheta_3\,\vartheta_{10}+\theta_3\,\theta_{10}\vartheta_7\,\vartheta_{14}$$
$$c_0c_8\,\Theta_5\,\Theta'_{13}=\theta_0\theta_8\,\vartheta_5\,\vartheta_{13}-\theta_5\,\theta_{13}\vartheta_0\,\vartheta_8-\theta_3\,\theta_{11}\vartheta_6\,\vartheta_{14}+\theta_6\,\theta_{14}\vartheta_3\,\vartheta_{11}$$
$$c_3c_8\,\Theta_6\,\Theta'_{13}=\theta_3\theta_8\,\vartheta_6\,\vartheta_{13}-\theta_6\,\theta_{13}\vartheta_3\,\vartheta_8-\theta_5\,\theta_{14}\vartheta_0\,\vartheta_{11}+\theta_0\,\theta_{11}\vartheta_5\,\vartheta_{14}$$
$$c_2c_8\,\Theta_7\,\Theta'_{13}=\theta_2\theta_8\,\vartheta_7\,\vartheta_{13}-\theta_7\,\theta_{13}\vartheta_2\,\vartheta_8-\theta_0\,\theta_{10}\vartheta_5\,\vartheta_{15}+\theta_5\,\theta_{15}\vartheta_0\,\vartheta_{10}$$
$$c_1c_4\,\Theta_8\,\Theta'_{13}=\theta_1\theta_4\,\vartheta_8\,\vartheta_{13}-\theta_8\,\theta_{13}\vartheta_1\,\vartheta_4-\theta_{11}\theta_{14}\vartheta_2\,\vartheta_7\,+\theta_2\,\theta_7\vartheta_{11}\,\vartheta_{14}$$
$$c_0c_4\,\Theta_9\,\Theta'_{13}=\theta_0\theta_4\,\vartheta_9\,\vartheta_{13}-\theta_9\,\theta_{13}\vartheta_0\,\vartheta_4-\theta_2\,\theta_6\,\vartheta_{11}\vartheta_{15}+\theta_{11}\theta_{15}\vartheta_2\,\vartheta_6$$
$$c_3c_4\,\Theta_{10}\Theta'_{13}=\theta_3\theta_4\,\vartheta_{10}\vartheta_{13}-\theta_{10}\theta_{13}\vartheta_3\,\vartheta_4-\theta_2\,\theta_5\,\vartheta_{11}\vartheta_{12}+\theta_{11}\theta_{12}\vartheta_2\,\vartheta_5$$
$$c_0c_6\,\Theta_{11}\Theta'_{13}=\theta_0\theta_6\,\vartheta_{11}\vartheta_{13}-\theta_{11}\theta_{13}\vartheta_0\,\vartheta_6-\theta_1\,\theta_7\,\vartheta_{10}\vartheta_{12}+\theta_{10}\theta_{12}\vartheta_1\,\vartheta_7$$
$$c_8c_9\,\Theta_{12}\Theta'_{13}=\theta_8\theta_9\,\vartheta_{12}\vartheta_{13}-\theta_{12}\theta_{13}\vartheta_8\,\vartheta_9-\theta_{10}\theta_{11}\vartheta_{14}\vartheta_{15}+\theta_{14}\theta_{15}\vartheta_{10}\vartheta_{11}$$
$$c_8c_8\,\Theta_{13}\Theta'_{13}=\theta_8\theta_8\,\vartheta_{13}\vartheta_{13}-\theta_{13}\theta_{13}\vartheta_8\,\vartheta_8-\theta_{14}\theta_{14}\vartheta_{11}\vartheta_{11}+\theta_{11}\theta_{11}\vartheta_{14}\vartheta_{14}$$
$$c_0c_3\,\Theta_{14}\Theta'_{13}=\theta_0\theta_3\,\vartheta_{14}\vartheta_{13}-\theta_{14}\theta_{13}\vartheta_0\,\vartheta_3-\theta_4\,\theta_7\,\vartheta_9\,\vartheta_{10}+\theta_9\,\theta_{10}\vartheta_4\,\vartheta_7$$
$$c_0c_2\,\Theta_{15}\Theta'_{13}=\theta_0\theta_2\,\vartheta_{15}\vartheta_{13}-\theta_{15}\theta_{13}\vartheta_0\,\vartheta_2-\theta_5\,\theta_7\,\vartheta_8\,\vartheta_{10}+\theta_8\,\theta_{10}\vartheta_5\,\vartheta_7$$

54. FIFTEENTH Set, with Θ'_{14}.

$$c_2c_{12}\Theta_0\,\Theta'_{14}=\theta_2\theta_{12}\vartheta_0\,\vartheta_{14}-\theta_0\,\theta_{14}\vartheta_2\vartheta_{12}-\theta_5\,\theta_{11}\vartheta_7\vartheta_9\,+\theta_7\theta_9\,\vartheta_5\,\vartheta_{11}$$
$$c_0c_{15}\Theta_1\,\Theta'_{14}=\theta_0\theta_{15}\vartheta_1\,\vartheta_{14}-\theta_1\,\theta_{14}\vartheta_0\vartheta_{15}-\theta_5\,\theta_{10}\vartheta_4\vartheta_{11}+\theta_4\theta_{11}\vartheta_5\,\vartheta_{10}$$
$$c_0c_{12}\Theta_2\,\Theta'_{14}=\theta_0\theta_{12}\vartheta_2\,\vartheta_{14}-\theta_2\,\theta_{14}\vartheta_0\vartheta_{12}-\theta_{11}\theta_7\,\vartheta_5\vartheta_9\,+\theta_5\theta_9\,\vartheta_7\,\vartheta_{11}$$
$$c_4c_9\,\Theta_3\,\Theta'_{14}=\theta_4\theta_9\,\vartheta_3\,\vartheta_{14}-\theta_3\,\theta_{14}\vartheta_4\vartheta_9\,-\theta_7\,\theta_{10}\vartheta_0\vartheta_{13}+\theta_0\theta_{13}\vartheta_7\,\vartheta_{10}$$
$$c_3c_9\,\Theta_4\,\Theta'_{14}=\theta_3\theta_9\,\vartheta_4\,\vartheta_{14}-\theta_4\,\theta_{14}\vartheta_3\vartheta_9\,-\theta_7\,\theta_{13}\vartheta_0\vartheta_{10}+\theta_0\theta_{10}\vartheta_7\,\vartheta_{13}$$
$$c_3c_8\,\Theta_5\,\Theta'_{14}=\theta_3\theta_8\,\vartheta_5\,\vartheta_{14}-\theta_5\,\theta_{14}\vartheta_3\vartheta_8\,-\theta_1\,\theta_{10}\vartheta_7\vartheta_{12}+\theta_7\theta_{12}\vartheta_1\,\vartheta_{10}$$
$$c_0c_8\,\Theta_6\,\Theta'_{14}=\theta_0\theta_8\,\vartheta_6\,\vartheta_{14}-\theta_6\,\theta_{14}\vartheta_0\vartheta_8\,-\theta_5\,\theta_{13}\vartheta_3\vartheta_{11}+\theta_3\theta_{11}\vartheta_5\,\vartheta_{13}$$
$$c_0c_9\,\Theta_7\,\Theta'_{14}=\theta_0\theta_9\,\vartheta_7\,\vartheta_{14}-\theta_7\,\theta_{14}\vartheta_0\vartheta_9\,-\theta_2\,\theta_{11}\vartheta_5\vartheta_{12}+\theta_5\theta_{12}\vartheta_2\,\vartheta_{11}$$
$$c_0c_6\,\Theta_8\,\Theta'_{14}=\theta_0\theta_6\,\vartheta_8\,\vartheta_{14}-\theta_8\,\theta_{14}\vartheta_0\vartheta_6\,-\theta_{11}\theta_{13}\vartheta_3\vartheta_5\,+\theta_3\theta_5\,\vartheta_{11}\vartheta_{13}$$
$$c_3c_4\,\Theta_9\,\Theta'_{14}=\theta_3\theta_4\,\vartheta_9\,\vartheta_{14}-\theta_9\,\theta_{14}\vartheta_3\vartheta_4\,-\theta_{10}\theta_{13}\vartheta_0\vartheta_7\,+\theta_0\theta_7\,\vartheta_{10}\vartheta_{13}$$
$$c_0c_4\,\Theta_{10}\Theta'_{14}=\theta_0\theta_4\,\vartheta_{10}\vartheta_{14}-\theta_{10}\theta_{14}\vartheta_0\vartheta_4\,-\theta_3\,\theta_7\,\vartheta_9\vartheta_{13}+\theta_9\theta_{13}\vartheta_3\,\vartheta_7$$
$$c_1c_4\,\Theta_{11}\Theta'_{14}=\theta_1\theta_4\,\vartheta_{11}\vartheta_{14}-\theta_{11}\theta_{14}\vartheta_1\vartheta_4\,-\theta_{10}\theta_{15}\vartheta_0\vartheta_5\,+\theta_0\theta_5\,\vartheta_{10}\vartheta_{15}$$
$$c_0c_2\,\Theta_{12}\Theta'_{14}=\theta_0\theta_2\,\vartheta_{12}\vartheta_{14}-\theta_{12}\theta_{14}\vartheta_0\vartheta_2\,-\theta_5\,\theta_7\,\vartheta_9\vartheta_{11}+\theta_9\theta_{11}\vartheta_5\,\vartheta_7$$
$$c_0c_3\,\Theta_{13}\Theta'_{14}=\theta_0\theta_3\,\vartheta_{13}\vartheta_{14}-\theta_{13}\theta_{14}\vartheta_0\vartheta_3\,-\theta_9\,\theta_{10}\vartheta_4\vartheta_7\,+\theta_4\theta_7\,\vartheta_9\,\vartheta_{10}$$
$$c_4c_4\,\Theta_{14}\Theta'_{14}=\theta_4\theta_4\,\vartheta_{14}\vartheta_{14}-\theta_{14}\theta_{14}\vartheta_4\vartheta_4\,-\theta_{13}\theta_{13}\vartheta_7\vartheta_7\,+\theta_7\theta_7\,\vartheta_{13}\vartheta_{13}$$
$$c_0c_1\,\Theta_{15}\Theta'_{14}=\theta_0\theta_1\,\vartheta_{15}\vartheta_{14}-\theta_{14}\theta_{15}\vartheta_0\vartheta_1\,-\theta_{10}\theta_{11}\vartheta_4\vartheta_5\,+\theta_4\theta_5\,\vartheta_{10}\vartheta_{11}$$

55. SIXTEENTH Set, with Θ'_{15}.

$$c_6 c_9 \Theta_0 \Theta'_{15} = \theta_6\theta_9\vartheta_0\vartheta_{15} + \theta_5\theta_{10}\vartheta_{12}\vartheta_3 - \theta_4\theta_{11}\vartheta_2\vartheta_{13} - \theta_7\theta_8\vartheta_1\vartheta_{14}$$
$$c_6 c_8 \Theta_1 \Theta'_{15} = \theta_6\theta_8\vartheta_1\vartheta_{15} + \theta_5\theta_{11}\vartheta_2\vartheta_{12} - \theta_0\theta_{14}\vartheta_7\vartheta_9 - \theta_3\theta_{13}\vartheta_4\vartheta_{10}$$
$$c_1 c_{12} \Theta_2 \Theta'_{15} = \theta_1\theta_{12}\vartheta_2\vartheta_{15} + \theta_7\theta_{10}\vartheta_4\vartheta_9 - \theta_6\theta_{11}\vartheta_5\vartheta_8 - \theta_0\theta_{13}\vartheta_3\vartheta_{14}$$
$$c_0 c_{12} \Theta_3 \Theta'_{15} = \theta_0\theta_{12}\vartheta_3\vartheta_{15} + \theta_7\theta_{11}\vartheta_4\vartheta_8 - \theta_5\theta_9\vartheta_{10}\vartheta_6 - \theta_2\theta_{14}\vartheta_1\vartheta_{13}$$
$$c_2 c_9 \Theta_4 \Theta'_{15} = \theta_2\theta_9\vartheta_4\vartheta_{15} + \theta_5\theta_{14}\vartheta_3\vartheta_8 - \theta_7\theta_{12}\vartheta_1\vartheta_{10} - \theta_0\theta_{11}\vartheta_6\vartheta_{13}$$
$$c_2 c_8 \Theta_5 \Theta'_{15} = \theta_2\theta_8\vartheta_5\vartheta_{15} + \theta_5\theta_{15}\vartheta_2\vartheta_8 - \theta_7\theta_{13}\vartheta_0\vartheta_{10} - \theta_0\theta_{10}\vartheta_7\vartheta_{13}$$
$$c_0 c_9 \Theta_6 \Theta'_{15} = \theta_0\theta_9\vartheta_6\vartheta_{15} + \theta_7\theta_{14}\vartheta_1\vartheta_8 - \theta_4\theta_{13}\vartheta_2\vartheta_{11} - \theta_3\theta_{10}\vartheta_5\vartheta_{12}$$
$$c_0 c_8 \Theta_7 \Theta'_{15} = \theta_0\theta_8\vartheta_7\vartheta_{15} + \theta_7\theta_{15}\vartheta_0\vartheta_8 - \theta_5\theta_{13}\vartheta_2\vartheta_{10} - \theta_2\theta_{10}\vartheta_5\vartheta_{13}$$
$$c_3 c_4 \Theta_8 \Theta'_{15} = \theta_3\theta_4\vartheta_8\vartheta_{15} + \theta_{10}\theta_{13}\vartheta_1\vartheta_6 - \theta_9\theta_{14}\vartheta_2\vartheta_5 - \theta_0\theta_7\vartheta_{11}\vartheta_{12}$$
$$c_0 c_6 \Theta_9 \Theta'_{15} = \theta_0\theta_6\vartheta_9\vartheta_{15} + \theta_{11}\theta_{13}\vartheta_2\vartheta_4 - \theta_8\theta_{14}\vartheta_1\vartheta_7 - \theta_3\theta_5\vartheta_{10}\vartheta_{12}$$
$$c_1 c_4 \Theta_{10} \Theta'_{15} = \theta_1\theta_4\vartheta_{10}\vartheta_{15} + \theta_{10}\theta_{15}\vartheta_1\vartheta_4 - \theta_{11}\theta_{14}\vartheta_0\vartheta_5 - \theta_0\theta_5\vartheta_{11}\vartheta_{14}$$
$$c_0 c_4 \Theta_{11} \Theta'_{15} = \theta_0\theta_4\vartheta_{11}\vartheta_{15} + \theta_{11}\theta_{15}\vartheta_0\vartheta_4 - \theta_{10}\theta_{14}\vartheta_1\vartheta_5 - \theta_1\theta_5\vartheta_{10}\vartheta_{14}$$
$$c_0 c_3 \Theta_{12} \Theta'_{15} = \theta_0\theta_3\vartheta_{12}\vartheta_{15} + \theta_{13}\theta_{14}\vartheta_1\vartheta_2 - \theta_8\theta_{11}\vartheta_7\vartheta_4 - \theta_5\theta_6\vartheta_9\vartheta_{10}$$
$$c_0 c_2 \Theta_{13} \Theta'_{15} = \theta_0\theta_2\vartheta_{13}\vartheta_{15} + \theta_{13}\theta_{15}\vartheta_0\vartheta_2 - \theta_5\theta_7\vartheta_8\vartheta_{10} - \theta_8\theta_{10}\vartheta_5\vartheta_7$$
$$c_0 c_1 \Theta_{14} \Theta'_{15} = \theta_0\theta_1\vartheta_{14}\vartheta_{15} + \theta_{14}\theta_{15}\vartheta_0\vartheta_0 - \theta_{10}\theta_{11}\vartheta_4\vartheta_5 - \theta_4\theta_5\vartheta_{11}\vartheta_{10}$$
$$c_{15} c_{15} \Theta_{15} \Theta'_{15} = \theta_{15}\theta_{15}\vartheta_{15}\vartheta_{15} + \theta_{10}\theta_{10}\vartheta_{10}\vartheta_{10} - \theta_{11}\theta_{11}\vartheta_{11}\vartheta_{11} - \theta_{14}\theta_{14}\vartheta_{14}\vartheta_{14}$$

56. Any one equation giving $cc\Theta_r\Theta'_n$ where r, n are different, also gives $cc\Theta_n\Theta'_r$ by changing the sign of ξ, η: thus from

$$c_2c_3\Theta_1\Theta'_0 = \theta_2\theta_3\vartheta_1\vartheta_0 + \theta_6\theta_7\vartheta_4\vartheta_5 - \theta_{10}\theta_{11}\vartheta_8\vartheta_9 - \theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}$$

there follows in this manner

$$c_2c_3\Theta_0\Theta'_1 = \theta_2\theta_3\vartheta_0\vartheta_1 - \theta_6\theta_7\vartheta_4\vartheta_5 - \theta_{10}\theta_{11}\vartheta_8\vartheta_9 + \theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}.$$

In addition to the equations given in formulæ (205), (206), (207), the following include most of those used in order to obtain the above sixteen sets in the form in which they are presented.

$$\left.\begin{aligned}
\theta_2\theta_3\vartheta_1\vartheta_0+\theta_8\theta_9\vartheta_{10}\vartheta_{11}&=\theta_0\theta_1\vartheta_2\vartheta_3+\theta_{10}\theta_{11}\vartheta_8\vartheta_9\\
\theta_4\theta_5\vartheta_6\vartheta_7+\theta_{14}\theta_{15}\vartheta_{12}\vartheta_{13}&=\theta_6\theta_7\vartheta_4\vartheta_5+\theta_{12}\theta_{13}\vartheta_{14}\vartheta_{15}\\
\theta_1\theta_3\vartheta_2\vartheta_0+\theta_4\theta_6\vartheta_5\vartheta_7&=\theta_5\theta_7\vartheta_4\vartheta_6+\theta_0\theta_2\vartheta_1\vartheta_3\\
\theta_{13}\theta_{15}\vartheta_{12}\vartheta_{14}+\theta_8\theta_{10}\vartheta_9\vartheta_{11}&=\theta_{12}\theta_{14}\vartheta_{13}\vartheta_{15}+\theta_9\theta_{11}\vartheta_8\vartheta_{10}\\
\theta_1\theta_2\vartheta_3\vartheta_0+\theta_{13}\theta_{14}\vartheta_{12}\vartheta_{15}&=\theta_0\theta_3\vartheta_1\vartheta_2+\theta_{12}\theta_{15}\vartheta_{13}\vartheta_{14}\\
\theta_4\theta_7\vartheta_5\vartheta_6+\theta_8\theta_{11}\vartheta_9\vartheta_{10}&=\theta_5\theta_6\vartheta_4\vartheta_7+\theta_9\theta_{10}\vartheta_8\vartheta_{11}
\end{aligned}\right\}\quad . \quad . \quad . \quad (208)$$

$$\left.\begin{aligned}
\theta_8\theta_{12}\vartheta_4\vartheta_0+\theta_{10}\theta_{14}\vartheta_2\vartheta_6&=\theta_0\theta_4\vartheta_8\vartheta_{12}+\theta_2\theta_6\vartheta_{10}\vartheta_{14}\\
\theta_9\theta_{13}\vartheta_1\vartheta_5+\theta_{11}\theta_{15}\vartheta_7\vartheta_3&=\theta_1\theta_5\vartheta_9\vartheta_{13}+\theta_7\theta_3\vartheta_{11}\vartheta_{15}\\
\theta_4\theta_{12}\vartheta_8\vartheta_0+\theta_5\theta_{13}\vartheta_1\vartheta_9&=\theta_0\theta_8\vartheta_4\vartheta_{12}+\theta_1\theta_9\vartheta_5\vartheta_{13}\\
\theta_2\theta_{10}\vartheta_6\vartheta_{14}+\theta_{11}\theta_3\vartheta_7\vartheta_{15}&=\theta_6\theta_{14}\vartheta_2\vartheta_{10}+\theta_7\theta_{15}\vartheta_3\vartheta_{11}\\
\theta_4\theta_8\vartheta_{12}\vartheta_0+\theta_{15}\theta_3\vartheta_7\vartheta_{11}&=\theta_0\theta_{12}\vartheta_4\vartheta_8+\theta_7\theta_{11}\vartheta_3\vartheta_{15}\\
\theta_{10}\theta_6\vartheta_2\vartheta_{14}+\theta_5\theta_9\vartheta_1\vartheta_{13}&=\theta_1\theta_{13}\vartheta_5\vartheta_9+\theta_2\theta_{14}\vartheta_6\vartheta_{10}
\end{aligned}\right\}\quad . \quad . \quad . \quad (209)$$

$$\left.\begin{aligned}
\theta_9\theta_{15}\vartheta_6\vartheta_0+\theta_4\theta_2\vartheta_{11}\vartheta_{13}&=\theta_0\theta_6\vartheta_9\vartheta_{15}+\theta_{11}\theta_{13}\vartheta_2\vartheta_4\\
\theta_3\theta_5\vartheta_{10}\vartheta_{12}+\theta_8\theta_{14}\vartheta_1\vartheta_7&=\theta_1\theta_7\vartheta_8\vartheta_{14}+\theta_{10}\theta_{12}\vartheta_3\vartheta_5\\
\theta_6\theta_{15}\vartheta_9\vartheta_0+\theta_8\theta_1\vartheta_{14}\vartheta_7&=\theta_{14}\theta_7\vartheta_1\vartheta_8+\theta_9\theta_0\vartheta_6\vartheta_{15}\\
\theta_2\theta_{11}\vartheta_4\vartheta_{13}+\theta_5\theta_{12}\vartheta_3\vartheta_{10}&=\theta_3\theta_{10}\vartheta_5\vartheta_{12}+\theta_4\theta_{13}\vartheta_2\vartheta_{11}\\
\theta_6\theta_9\vartheta_{15}\vartheta_0+\theta_5\theta_{10}\vartheta_3\vartheta_{12}&=\theta_0\theta_{15}\vartheta_6\vartheta_9+\theta_3\theta_{12}\vartheta_5\vartheta_{10}\\
\theta_8\theta_7\vartheta_1\vartheta_{14}+\theta_4\theta_{11}\vartheta_2\vartheta_{13}&=\theta_2\theta_{13}\vartheta_4\vartheta_{11}+\theta_1\theta_{14}\vartheta_7\vartheta_8
\end{aligned}\right\}\quad . \quad . \quad . \quad (210)$$

$$\left.\begin{aligned}
\theta_1\theta_9\vartheta_0\vartheta_8+\theta_4\theta_{12}\vartheta_5\vartheta_{13}&=\theta_0\theta_8\vartheta_1\vartheta_9+\theta_5\theta_{13}\vartheta_4\vartheta_{12}\\
\theta_{11}\theta_3\vartheta_2\vartheta_{10}+\theta_6\theta_{14}\vartheta_7\vartheta_{15}&=\theta_2\theta_{10}\vartheta_3\vartheta_{11}+\theta_7\theta_{15}\vartheta_6\vartheta_{14}\\
\theta_1\theta_8\vartheta_0\vartheta_9+\theta_6\theta_{15}\vartheta_7\vartheta_{14}&=\theta_0\theta_9\vartheta_1\vartheta_8+\theta_7\theta_{14}\vartheta_6\vartheta_{15}\\
\theta_{11}\theta_2\vartheta_3\vartheta_{10}+\theta_5\theta_{12}\vartheta_4\vartheta_{13}&=\theta_3\theta_{10}\vartheta_2\vartheta_{11}+\theta_4\theta_{13}\vartheta_5\vartheta_{12}\\
\theta_8\theta_9\vartheta_0\vartheta_1+\theta_{10}\theta_{11}\vartheta_2\vartheta_3&=\theta_0\theta_1\vartheta_8\vartheta_9+\theta_2\theta_3\vartheta_{10}\vartheta_{11}\\
\theta_{12}\theta_{13}\vartheta_4\vartheta_5+\theta_{14}\theta_{15}\vartheta_6\vartheta_7&=\theta_4\theta_5\vartheta_{12}\vartheta_{13}+\theta_6\theta_7\vartheta_{14}\vartheta_{15}
\end{aligned}\right\}\quad . \quad . \quad . \quad (211)$$

$$\left.\begin{aligned}
\theta_0\theta_{12}\vartheta_{14}\vartheta_2+\theta_3\theta_{15}\vartheta_{13}\vartheta_1&=\theta_4\theta_8\vartheta_6\vartheta_{10}+\theta_{11}\theta_7\vartheta_5\vartheta_9\\
\theta_0\theta_2\vartheta_{14}\vartheta_{12}+\theta_3\theta_1\vartheta_{13}\vartheta_{15}&=\theta_4\theta_6\vartheta_8\vartheta_{10}+\theta_5\theta_7\vartheta_9\vartheta_{11}\\
\theta_2\theta_{12}\vartheta_{14}\vartheta_0+\theta_{15}\theta_1\vartheta_3\vartheta_{13}&=\theta_6\theta_8\vartheta_4\vartheta_{10}+\theta_5\theta_{11}\vartheta_7\vartheta_9
\end{aligned}\right\}\quad . \quad . \quad . \quad (212)$$

$$\left.\begin{aligned}
\theta_0\theta_6\vartheta_7\vartheta_1+\theta_{11}\theta_{13}\vartheta_{10}\vartheta_{12}&=\theta_9\theta_{15}\vartheta_8\vartheta_{14}+\theta_2\theta_4\vartheta_3\vartheta_5\\
\theta_0\theta_1\vartheta_7\vartheta_6+\theta_{10}\theta_{11}\vartheta_{12}\vartheta_{13}&=\theta_8\theta_9\vartheta_{14}\vartheta_{15}+\theta_2\theta_3\vartheta_4\vartheta_5\\
\theta_1\theta_6\vartheta_7\vartheta_0+\theta_{10}\theta_{13}\vartheta_{12}\vartheta_{11}&=\theta_8\theta_{15}\vartheta_9\vartheta_{14}+\theta_4\theta_3\vartheta_2\vartheta_5
\end{aligned}\right\}\quad . \quad . \quad . \quad (213)$$

$$\left.\begin{aligned}
\theta_0\theta_3\vartheta_{11}\vartheta_8+\theta_{13}\theta_{14}\vartheta_5\vartheta_6&=\theta_{12}\theta_{15}\vartheta_4\vartheta_7+\theta_1\theta_2\vartheta_9\vartheta_{10}\\
\theta_3\theta_8\vartheta_{11}\vartheta_0+\theta_5\theta_{14}\vartheta_{13}\vartheta_6&=\theta_4\theta_{15}\vartheta_{12}\vartheta_7+\theta_2\theta_9\vartheta_1\vartheta_{10}\\
\theta_0\theta_8\vartheta_{11}\vartheta_3+\theta_5\theta_{13}\vartheta_6\vartheta_{14}&=\theta_4\theta_{12}\vartheta_7\vartheta_{15}+\theta_1\theta_9\vartheta_2\vartheta_{10}
\end{aligned}\right\}\quad\ldots\quad(214)$$

$$\left.\begin{aligned}
\theta_0\theta_8\vartheta_{10}\vartheta_2+\theta_5\theta_{13}\vartheta_7\vartheta_{15}&=\theta_4\theta_{12}\vartheta_6\vartheta_{14}+\theta_1\theta_9\vartheta_3\vartheta_{11}\\
\theta_2\theta_8\vartheta_{10}\vartheta_0+\theta_7\theta_{13}\vartheta_5\vartheta_{15}&=\theta_6\theta_{12}\vartheta_4\vartheta_{14}+\theta_3\theta_9\vartheta_1\vartheta_{11}\\
\theta_0\theta_2\vartheta_{10}\vartheta_8+\theta_5\theta_7\vartheta_{13}\vartheta_{15}&=\theta_4\theta_6\vartheta_{12}\vartheta_{14}+\theta_1\theta_3\vartheta_9\vartheta_{11}
\end{aligned}\right\}\quad\ldots\quad(215)$$

$$\left.\begin{aligned}
\theta_2\theta_6\vartheta_5\vartheta_1+\theta_8\theta_{12}\vartheta_{11}\vartheta_{15}&=\theta_{10}\theta_{14}\vartheta_9\vartheta_{13}+\theta_0\theta_4\vartheta_3\vartheta_7\\
\theta_1\theta_6\vartheta_5\vartheta_2+\theta_8\theta_{15}\vartheta_{11}\vartheta_{12}&=\theta_{10}\theta_{13}\vartheta_9\vartheta_{14}+\theta_3\theta_4\vartheta_0\vartheta_7\\
\theta_1\theta_2\vartheta_5\vartheta_6+\theta_{12}\theta_{15}\vartheta_{11}\vartheta_8&=\theta_{13}\theta_{14}\vartheta_9\vartheta_{10}+\theta_0\theta_3\vartheta_4\vartheta_7
\end{aligned}\right\}\quad\ldots\quad(216)$$

$$\left.\begin{aligned}
\theta_4\theta_6\vartheta_1\vartheta_3+\theta_5\theta_7\vartheta_0\vartheta_2&=\theta_1\theta_3\vartheta_4\vartheta_6+\theta_0\theta_2\vartheta_5\vartheta_7\\
\theta_9\theta_{11}\vartheta_{12}\vartheta_{14}+\theta_8\theta_{10}\vartheta_{13}\vartheta_{15}&=\theta_{12}\theta_{14}\vartheta_9\vartheta_{11}+\theta_{13}\theta_{15}\vartheta_8\vartheta_{10}\\
\theta_3\theta_4\vartheta_1\vartheta_6+\theta_8\theta_{15}\vartheta_{10}\vartheta_{13}&=\theta_1\theta_6\vartheta_4\vartheta_3+\theta_{10}\theta_{13}\vartheta_8\vartheta_{15}\\
\theta_2\theta_5\vartheta_0\vartheta_7+\theta_9\theta_{14}\vartheta_{11}\vartheta_{12}&=\theta_0\theta_7\vartheta_2\vartheta_5+\theta_{11}\theta_{12}\vartheta_9\vartheta_{14}\\
\theta_3\theta_6\vartheta_1\vartheta_4+\theta_9\theta_{12}\vartheta_{11}\vartheta_{14}&=\theta_1\theta_4\vartheta_3\vartheta_6+\theta_{11}\theta_{14}\vartheta_9\vartheta_{12}\\
\theta_2\theta_7\vartheta_0\vartheta_5+\theta_8\theta_{13}\vartheta_{15}\vartheta_{10}&=\theta_0\theta_5\vartheta_2\vartheta_7+\theta_{10}\theta_{15}\vartheta_8\vartheta_{13}
\end{aligned}\right\}\quad\ldots\quad(217)$$

$$\left.\begin{aligned}
\theta_0\theta_3\vartheta_{15}\vartheta_{12}+\theta_{13}\theta_{14}\vartheta_1\vartheta_2&=\theta_{15}\theta_{12}\vartheta_0\vartheta_3+\theta_1\theta_2\vartheta_{13}\vartheta_{14}\\
\theta_3\theta_{12}\vartheta_{15}\vartheta_0+\theta_5\theta_{10}\vartheta_9\vartheta_6&=\theta_{15}\theta_0\vartheta_3\vartheta_{12}+\theta_9\theta_6\vartheta_5\vartheta_{10}\\
\theta_0\theta_{12}\vartheta_{15}\vartheta_3+\theta_{11}\theta_7\vartheta_4\vartheta_8&=\theta_4\theta_8\vartheta_{11}\vartheta_7+\theta_{15}\theta_3\vartheta_0\vartheta_{12}\\
\theta_2\theta_{14}\vartheta_1\vartheta_{13}+\theta_9\theta_5\vartheta_{10}\vartheta_6&=\theta_1\theta_{13}\vartheta_2\vartheta_{14}+\theta_6\theta_{10}\vartheta_9\vartheta_5\\
\theta_{11}\theta_4\vartheta_7\vartheta_8+\theta_{13}\theta_2\vartheta_1\vartheta_{14}&=\theta_7\theta_8\vartheta_4\vartheta_{11}+\theta_1\theta_{14}\vartheta_2\vartheta_{13}\\
\theta_8\theta_{11}\vartheta_4\vartheta_7+\theta_5\theta_6\vartheta_9\vartheta_{10}&=\theta_4\theta_7\vartheta_8\vartheta_{11}+\theta_9\theta_{10}\vartheta_5\vartheta_6
\end{aligned}\right\}\quad\ldots\quad(218)$$

$$\left.\begin{aligned}
\theta_9\theta_4\vartheta_{14}\vartheta_3+\theta_2\theta_{15}\vartheta_5\vartheta_8&=\theta_1\theta_{12}\vartheta_6\vartheta_{11}+\theta_7\theta_{10}\vartheta_0\vartheta_{13}\\
\theta_3\theta_9\vartheta_{14}\vartheta_4+\theta_2\theta_8\vartheta_5\vartheta_{15}&=\theta_6\theta_{12}\vartheta_1\vartheta_{11}+\theta_7\theta_{13}\vartheta_0\vartheta_{10}\\
\theta_3\theta_4\vartheta_{14}\vartheta_9+\theta_8\theta_{15}\vartheta_5\vartheta_2&=\theta_1\theta_6\vartheta_{12}\vartheta_{11}+\theta_{10}\theta_{13}\vartheta_0\vartheta_7
\end{aligned}\right\}\quad\ldots\quad(219)$$

and others, as follow, which do not admit of being arranged symmetrically, the equations necessary to complete systems such as (218) or (219) not having been used;

$$
\begin{aligned}
\theta_1\theta_4\vartheta_0\vartheta_5+\theta_{14}\theta_{11}\vartheta_{10}\vartheta_{15}&=\theta_3\theta_6\vartheta_2\vartheta_7+\theta_9\theta_{12}\vartheta_8\vartheta_{13}\\
\theta_1\theta_0\vartheta_4\vartheta_5+\theta_{10}\theta_{11}\vartheta_{14}\vartheta_{15}&=\theta_3\theta_2\vartheta_6\vartheta_7+\theta_8\theta_9\vartheta_{12}\vartheta_{13}\\
\theta_1\theta_{12}\vartheta_{13}\vartheta_0+\theta_2\theta_{15}\vartheta_{14}\vartheta_3&=\theta_4\theta_9\vartheta_8\vartheta_5+\theta_{10}\theta_7\vartheta_6\vartheta_{11}\\
\theta_0\theta_{12}\vartheta_{13}\vartheta_1+\theta_3\theta_{15}\vartheta_{14}\vartheta_2&=\theta_4\theta_8\vartheta_5\vartheta_9+\theta_{11}\theta_7\vartheta_6\vartheta_{10}\\
\theta_3\theta_8\vartheta_{10}\vartheta_1+\theta_4\theta_{15}\vartheta_{13}\vartheta_6&=\theta_2\theta_9\vartheta_0\vartheta_{11}+\theta_5\theta_{14}\vartheta_{12}\vartheta_7\\
\theta_1\theta_8\vartheta_{10}\vartheta_3+\theta_6\theta_{15}\vartheta_{13}\vartheta_4&=\theta_0\theta_9\vartheta_2\vartheta_{11}+\theta_7\theta_{14}\vartheta_{12}\vartheta_5\\
\theta_0\theta_{15}\vartheta_{14}\vartheta_1+\theta_3\theta_{12}\vartheta_2\vartheta_{13}&=\theta_6\theta_9\vartheta_8\vartheta_7+\theta_5\theta_{10}\vartheta_4\vartheta_{11}\\
\theta_0\theta_1\vartheta_{14}\vartheta_{15}+\theta_2\theta_3\vartheta_{12}\vartheta_{13}&=\theta_8\theta_9\vartheta_6\vartheta_7+\theta_{10}\theta_{11}\vartheta_4\vartheta_5\\
\theta_0\theta_9\vartheta_{11}\vartheta_2+\theta_7\theta_{14}\vartheta_{12}\vartheta_5&=\theta_1\theta_8\vartheta_3\vartheta_{10}+\theta_6\theta_{15}\vartheta_4\vartheta_{13}\\
\theta_0\theta_2\vartheta_{11}\vartheta_9+\theta_7\theta_5\vartheta_{12}\vartheta_{14}&=\theta_1\theta_3\vartheta_8\vartheta_{10}+\theta_6\theta_4\vartheta_{15}\vartheta_{13}\\
\theta_0\theta_4\vartheta_3\vartheta_7+\theta_{10}\theta_{14}\vartheta_9\vartheta_{13}&=\theta_8\theta_{12}\vartheta_{11}\vartheta_{15}+\theta_2\theta_6\vartheta_1\vartheta_5\\
\theta_0\theta_3\vartheta_4\vartheta_7+\theta_{13}\theta_{14}\vartheta_9\vartheta_{10}&=\theta_{15}\theta_{12}\vartheta_{11}\vartheta_8+\theta_2\theta_1\vartheta_6\vartheta_5
\end{aligned}
$$

and many of a similar form.

57. If there be four pairs of arguments x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, such that

$$x_1+x_2+x_3+x_4=0=y_1+y_2+y_3+y_4$$

then with the notation of the first section we have

$$
\begin{aligned}
X_1+x_1=X_2+x_2=X_3+x_3=X_4+x_4=0\\
Y_1+y_1=Y_2+y_2=Y_3+y_3=Y_4+y_4=0
\end{aligned}
$$

and the product theorem (23) will give results similar to (205), (206), (207). As examples, if we put

$$\Pi=\vartheta(x_1,\ y_1)\vartheta(x_2,\ y_2)\vartheta(x_3,\ y_3)\vartheta(x_4,\ y_4)$$

the equations corresponding to (205) will be

$$
\begin{aligned}
\Pi_0+\Pi_5+\Pi_{14}&=\Pi_1+\Pi_6+\Pi_{12}\\
\Pi_0+\Pi_{10}+\Pi_{13}&=\Pi_2+\Pi_9+\Pi_{12}
\end{aligned}
$$

and so on ; and if

$$\Pi_s^r=\vartheta_r(x_1,\ y_1)\vartheta_r(x_2,\ y_2)\vartheta_s(x_3,\ y_3)\vartheta_s(x_4,\ y_4)$$

the first equations of the sets corresponding to (206), (207) will be respectively

$$
\begin{aligned}
\Pi_5^0+\Pi_7^2+\Pi_6^3+\Pi_4^1&=\Pi_0^5+\Pi_2^7+\Pi_3^6+\Pi_1^4\\
\Pi_2^8+\Pi_6^{12}+\Pi_4^{14}+\Pi_0^{10}&=\Pi_8^2+\Pi_{12}^6+\Pi_{14}^4+\Pi_{10}^0
\end{aligned}
$$

58. Again, if

$$a+b+c+d=0$$
$$a'+b'+c'+d'=0$$

and in the general theorem we put

$$x_1=u+a \qquad x_2=u+b \qquad x_3=u+c \qquad x_4=u+d$$
$$y_1=v+a' \qquad y_2=v+b' \qquad y_3=v+c' \qquad y_4=v+d'$$

then $X_1=u-a$, $Y_1=v-a'$, and so for the others: and it is not difficult to prove that

$$\begin{aligned}&\vartheta_0(u+a)\vartheta_1(u+b)\vartheta_2(u+c)\vartheta_3(u+d)+\vartheta_2(u+a)\vartheta_3(u+b)\vartheta_0(u+c)\vartheta_1(u+d)\\&\qquad+\vartheta_0(u-a)\vartheta_1(u-b)\vartheta_2(u-c)\vartheta_3(u-d)+\vartheta_2(u-a)\vartheta_3(u-b)\vartheta_0(a-c)\vartheta_1(u-d)\\&=\vartheta_3(u+a)\vartheta_2(u+b)\vartheta_1(u+c)\vartheta_0(u+d)+\vartheta_1(u+a)\vartheta_0(u+b)\vartheta_3(u+c)\vartheta_2(u+d)\\&\qquad+\vartheta_3(u-a)\vartheta_2(u-b)\vartheta_1(u-c)\vartheta_0(u-d)+\vartheta_1(u-a)\vartheta_0(u-b)\vartheta_3(u-c)\vartheta_2(u+d)\end{aligned}$$

$\vartheta(u+a), \ldots$ denoting $\vartheta(u+a,\ v+a'), \ldots$, with other relations of the same kind between the theta-functions.

SECTION IV.

The "r" tuple theta-functions.

59. The general "r" tuple theta-functions is defined by the equation

$$\begin{aligned}\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_r\right\}&=\Sigma\Sigma\ldots(-1)^{m_1\lambda_1+\ldots+m_r\lambda_r} p_1^{\left(m_1+\frac{\nu_1}{2}\right)^2} p_2^{\left(m_2+\frac{\nu_2}{2}\right)^2}\ldots p_r^{\left(m_r+\frac{\nu_r}{2}\right)^2}\\ &\quad p_{1,2}^{2\left(m_1+\frac{\nu_1}{2}\right)\left(m_2+\frac{\nu_2}{2}\right)}\ldots p_{s,t}^{2\left(m_s+\frac{\nu_s}{2}\right)\left(m_t+\frac{\nu_t}{2}\right)}\ldots v_1^{(2m_1+\nu_1)x_1} v_2^{(2m_2+\nu_2)x_2}\ldots v_r^{(2m_r+\nu_r)x_r} \qquad (220)\end{aligned}$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_r, \nu_1, \nu_2, \ldots, \nu_r$ are given integers (afterwards taken to be either zero or unity) and $\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix}$ is called the characteristic; $x_1, x_2, \ldots, x_r$ are the variables; $p_1, p_2, \ldots, p_r, p_{1,2}, \ldots, p_{s,t}, \ldots, v_1, v_2, \ldots, v_r$ are $\frac{r\,.\,r+3}{2}$ constants and are called the parameters; and the "r" tuple summation extends to all positive and negative integral values from $-\infty$ to $+\infty$ (including zero) of $m_1, m_2, \ldots, m_r$. To ensure the convergence of the series it is necessary that the real part of

$$(2m_1+\nu_1)^2\log p_1+\ldots+2(2m_1+\nu_1)(2m_2+\nu^2)\log p_{1,2}+\ldots$$

should be negative for all real values of the m's; beyond this restriction the $\frac{r.r+1}{2}$ quantities p are of any form or value whatever.

60. From the definition it at once follows that

$$\Phi\begin{Bmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_s, \quad \ldots, \nu_r\end{Bmatrix}=\Phi\begin{Bmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{Bmatrix}$$

$$=(-1)^{\lambda_s}\Phi\begin{Bmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \quad \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_s+2, \ldots, \nu_r\end{Bmatrix} \quad . \quad . \quad . \quad . \quad (221)$$

the variables being the same throughout. Hence it is obvious that the number of distinct functions is $2^{2r}=4^r$. Also

$$\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix}-x_1, -x_2, \ldots, -x_r\right\}$$

$$=(-1)^{\lambda_1\nu_1+\ldots+\lambda_r\nu_r}\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix}x_1, x_2, \ldots, x_r\right\} \quad . \quad . \quad . \quad (222)$$

a formula which enables us to distinguish between even and uneven functions. For each of the expressions, as $\lambda_s\nu_s$, zero value may arise in three ways, viz.: $\lambda_s=0$, $\nu_s=0$; $\lambda_s=1$, $\nu_s=0$; $\lambda_s=0$, $\nu_s=1$; and a value unity arises in one way, $\lambda_s=1$, $\nu_s=1$; and an uneven function will occur when the number of units in the index is odd. Thus if P, Q denote the numbers of even and uneven functions respectively

$$P=3^r+\frac{r.r-1}{1.2}3^{r-2}+\frac{r.r-1.r-2.r-3}{1.2.3.4}3^{r-4}+\ldots$$

$$Q=r.3^{r-1}+\frac{r.r-1.r-2}{1.2.3}3^{r-3}+\ldots$$

and therefore

$$P+Q=(3+1)^r=2^{2r}$$

$$P-Q=(3-1)^r=2^r$$

so that

$$P=2^{2r-1}+2^{r-1}$$

$$Q=2^{2r-1}-2^{r-1}.$$

Periodicity.

61. Putting

$$v_s = e^{\frac{i\pi}{2K_s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

where s has in turn every value from 1 to r, in the definition of Φ there follow at once the r distinct sets of actual periods for Φ :—

x_1	x_2	x_3				x_s	x_r
$4K_1$	0	0				0	0
0	$4K_2$	0				0	0
0	0	$4K_3$				0	0
0	0	0				$4K_s$	0
0	0	0				0	$4K_r$

and

$$\left.\begin{aligned}
&\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_s + K_s, \ldots, x_r\right\} \\
&\qquad = (-1)^{\frac{1}{2}\nu_s}\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s \quad, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_s, \ldots x_r\right\} \\
&\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_s + 2K_s, \ldots, x_r\right\} \\
&\qquad = (-1)^{\nu_s}\,\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_s, \ldots, x_r\right\}
\end{aligned}\right\} . \quad (224).$$

By a method similar to that before used can be obtained the following set of quasi-periods :—

x_1	x_2			x_s	x_r
$\frac{4K_1}{\pi i}\log p_1$	$\frac{4K_2}{\pi i}\log p_{1,2}$			$\frac{4K_s}{\pi i}\log p_{1,s}$	$\frac{4K_r}{\pi i}\log p_{1,r}$
$\frac{4K_1}{\pi i}\log p_{1,2}$	$\frac{4K_2}{\pi i}\log p_2$			$\frac{4K_s}{\pi i}\log p_{2,s}$	$\frac{4K_r}{\pi i}\log p_{2,r}$
$\frac{4K_1}{\pi i}\log p_{1,s}$	$\frac{4K_2}{\pi i}\log p_{2,s}$			$\frac{4K_s}{\pi i}\log p_s$	$\frac{4K_r}{\pi i}\log p_{s,r}$
$\frac{4K_1}{\pi i}\log p_{1,r}$	$\frac{4K_2}{\pi i}\log p_{2,r}$			$\frac{4K_s}{\pi i}\log p_{s,r}$	$\frac{4K_r}{\pi i}\log p_r$

and

$$\left.\begin{aligned}
&\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix} x_1+\frac{K_1}{\pi i}\log p_{1,s},\ x_2+\frac{K_2}{\pi i}\log p_{2,s}, \ldots, x_s\right.\\
&\qquad\qquad\left.+\frac{K_s}{\pi i}\log p_s, \ldots, x_r+\frac{K_r}{\pi i}\log p_{r,s}\right\}\\
&\quad=p_s^{-\frac{1}{4}}e^{-\frac{i\pi x_s}{2K_s}}\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s\quad, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s+1, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_s, \ldots, x_r\right\}\\
&\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix} x_1+\frac{2K_1}{\pi i}\log p_{1,s},\ x_2+\frac{2K_2}{\pi i}\log p_{2,s}, \ldots, x_s\right.\\
&\qquad\qquad\left.+\frac{2K_s}{\pi i}\log p_s, \ldots, x_r+\frac{2K_r}{\pi i}\log p_{s,r}\right\}\\
&\quad=p_s^{-1}e^{-\frac{i\pi x_s}{K_s}}(-1)^{\lambda_s}\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_s, \ldots, x_r\right\}
\end{aligned}\right\} \quad (225).$$

Product theorem.

62. We multiply four functions $\Phi\ \Phi'\ \Phi''\ \Phi'''$ (in which the variables are $x_1, \ldots,$ $x'_1, \ldots, x''_1, \ldots, x'''_1, \ldots,$ and the characteristics are $\begin{pmatrix}\lambda_1, \ldots \\ \nu_1, \ldots\end{pmatrix}, \begin{pmatrix}\lambda'_1, \ldots \\ \nu'_1, \ldots\end{pmatrix}, \begin{pmatrix}\lambda''_1, \ldots \\ \nu''_1, \ldots\end{pmatrix},$ and $\begin{pmatrix}\lambda'''_1, \ldots \\ \nu'''_1, \ldots\end{pmatrix}$ respectively, the sums of the four corresponding numbers being all even); and we find that the product is the sum of 4^r products. Denote such a product by

$$\Pi\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_r\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (226)$$

Let

$$\begin{aligned}
&\mathrm{M}_t+2m_t=\mathrm{M}'_t+2m'_t=\mathrm{M}''_t+2m''_t=\mathrm{M}'''_t+2m'''_t=m'_t+m'_t+m''_t+m''_t \\
&2(\mathrm{X}_t+x_t)=2(\mathrm{X}'_t+x'_t)=2(\mathrm{X}''_t+x''_t)=2(\mathrm{X}'''_t+x'''_t)=x_t+x'_t+x''_t+x'''_t \\
&2(\Lambda_t+\lambda_t)=2(\Lambda'_t+\lambda'_t)=2(\Lambda''_t+\lambda''_t)=2(\Lambda'''_t+\lambda'''_t)=\lambda_t+\lambda'_t+\lambda''_t+\lambda'''_t=2\mathrm{L}_t \\
&2(\mathrm{N}_t+\nu_t)=2(\mathrm{N}'_t+\nu'_t)=2(\mathrm{N}''_t+\nu''_t)=2(\mathrm{N}'''_t+\nu'''_t)=\nu_t+\nu'_t+\nu''_t+\nu'''_t
\end{aligned}$$

in which for t are to be substituted, in succession, the values 1, 2, 3, . . . , r. Then

$$m_t\lambda_t+m'_t\lambda'_t+m''_t\lambda''_t+m'''_t\lambda'''_t=\tfrac{1}{2}(\mathrm{M}_t\Lambda_t+\mathrm{M}'_t\Lambda'_t+\mathrm{M}''_t\Lambda''_t+\mathrm{M}'''_t\Lambda'''_t)$$

$$\begin{aligned}
&(2m_t+\nu_t)^2+(2m'_t+\nu'_t)^2+(2m''_t+\nu''_t)^2+(2m'''_t+\nu'''_t) \\
&\qquad\qquad=(\mathrm{M}_t+\mathrm{N}_t)^2+(\mathrm{M}'_t+\mathrm{N}'_t)^2+(\mathrm{M}''_t+\mathrm{N}''_t)^2+(\mathrm{M}'''_t+\mathrm{N}'''_t)
\end{aligned}$$

$$\begin{aligned}
&(2m_t+\nu_t)(2m_s+\nu_s)+\ldots+(2m'''_t+\nu'''_t)(2m'''_s+\nu'''_s) \\
&\qquad\qquad=(\mathrm{M}_t+\mathrm{N}_t)(\mathrm{M}_s+\mathrm{N}_s)+\ldots+(\mathrm{M}'''_t+\mathrm{N}'''_t)(\mathrm{M}'''_s+\mathrm{N}'''_s)
\end{aligned}$$

$$(m_t+\nu_t)x_t+\ldots+(2m'''_t+\nu'''_t)x'''_t=(\mathrm{M}_t+\mathrm{N}_t)\mathrm{X}_t+\ldots+(\mathrm{M}'''_t+\mathrm{N}'''_t)\mathrm{X}'''_t.$$

These, substituted in $\Phi\Phi'\Phi''\Phi'''$, give

$$\begin{aligned}
\Phi\Phi'\Phi''\Phi'''=\Sigma\Sigma\ldots(-1)^{\sum\limits_{t=1}^{t=r}\frac{\mathrm{M}_t\Lambda_t+\mathrm{M}'_t\Lambda'_t+\mathrm{M}''_t\Lambda''_t+\mathrm{M}'''_t\Lambda'''_t}{2}}\, & p_1^{\frac{1}{4}\{(\mathrm{M}_1+\mathrm{N}_1)^2+\ldots+(\mathrm{M}'''_1+\mathrm{N}'''_1)^2\}}\ldots \\
& p_r^{\frac{1}{4}\{(\mathrm{M}_r+\mathrm{N}_r)^2+\ldots+(\mathrm{M}'''_r+\mathrm{N}'''_r)^2\}}\, p_{1,2}^{\frac{1}{2}\{(\mathrm{M}_1+\mathrm{N}_1)(\mathrm{M}_2+\mathrm{N}_2)+\ldots+(\mathrm{M}'''_1+\mathrm{N}'''_1)(\mathrm{M}'''_2+\mathrm{N}'''_2)\}}\ldots \\
& p_{s,r}^{\frac{1}{2}\{(\mathrm{M}_s+\mathrm{N}_s)(\mathrm{M}_r+\mathrm{N}_r)+\ldots+(\mathrm{M}'''_s+\mathrm{N}'''_s)(\mathrm{M}'''_r+\mathrm{N}'''_r)\}}\, v_1^{(\mathrm{M}_1+\mathrm{N}_1)\mathrm{X}_1+\ldots+(\mathrm{M}'''_1+\mathrm{N}'''_1)\mathrm{X}'''_1}\ldots \\
& v_r^{(\mathrm{M}_r+\mathrm{N}_r)\mathrm{X}_r+\ldots+(\mathrm{M}'''_r+\mathrm{N}'''_r)\mathrm{X}'''_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (227)
\end{aligned}$$

the summation being taken for all values of the M's defined by the preceding equations. Now the difference between any two of the M's with the same suffix is even, so that all the M's with the same suffix are either even or uneven. In the former case let

$$\mathrm{M}_t=2\mu_t \qquad \mathrm{M}'_t=2\mu'_t \qquad \mathrm{M}''_t=2\mu''_t \qquad \mathrm{M}'''_t=2\mu'''_t$$

and it will be found that if the equations are satisfied

$$\mu_t+\mu'_t+\mu''_t+\mu'''_t=\text{even.}$$

In the latter case, let

$$\mathrm{M}_t=2\mu_t+1 \qquad \mathrm{M}'_t=2\mu'_t+1 \qquad \mathrm{M}''_t=2\mu''_t+1 \qquad \mathrm{M}'''_t=2\mu'''_t+1$$

and then it will be necessary that

$$\mu_t+\mu'_t+\mu''_t+\mu'''_t=\text{uneven.}$$

Separate now the general term in (227) into parts corresponding to the particular cases of the values of the M's (*i.e.*, whether they are even or uneven), and denote them as follows :—

Σ_0 when all the M's are even, and general term P_0:
Σ_t when all the M's except the M_t's are even, and general term P_t:
$\Sigma_{s,t}$ when all the M's except the M_s, M_t are even, and general term $P_{s,t}$:

and so on ; also let

$\overset{t=r}{\underset{t=1}{S}}\Sigma_t=$ sum of all the terms which have one set of M's uneven and all the rest even,

$\overset{s=r}{\underset{s=1}{S}}\overset{t=r}{\underset{t=1}{S}}\Sigma_{s,t}=$sum of all the terms which have two sets of M's uneven and all the rest even,

and so on ; making the number of distinct terms on the right-hand side

1	in which no sets of M are uneven
$+r$	in which one set is uneven
$+\dfrac{r.r-1}{2\,!}$	in which two sets are uneven
$+\dfrac{r.r-1.r-2}{3\,!}$	in which three sets are uneven
$+\ldots$	
$+1$	in which all the sets are uneven

viz.: $=2^r$ in all ; and then

$$\Phi\Phi'\Phi''\Phi'''=\Sigma_0 P_0+\overset{t=r}{\underset{t=1}{S}}(-1)^{L_t}\Sigma_t P_t+\overset{s=r}{\underset{s=1}{S}}\overset{t=r}{\underset{t=1}{S}}(-1)^{L_s+L_t}\Sigma_{s,t}P_{s,t}+\ldots \quad . \quad . \quad (228).$$

In this

$$P_0=(-1)^{\sum\limits_{t=1}^{t=r}(\mu_t\Lambda_t+\mu'_t\Lambda'_t+\mu''_t\Lambda''_t+\mu'''_t\Lambda'''_t)}\,p_1^{\frac{1}{4}\{(2\mu_1+N_1)^2+\ldots+(2\mu'''_1+N'''_1)^2\}}\ldots p_r^{\frac{1}{4}\{(2\mu_r+N_r)+\ldots+(2\mu'''_r+N'''_r)^2\}}$$
$$p_{1,2}^{\frac{1}{2}\{(2\mu_1+N_1)(2\mu_2+N_2)+\ldots+(2\mu'''_1+N'''_1)(2\mu'''_2+N'''_2)\}}\ldots$$
$$v_1^{(2\mu_1+N_1)X_1+\ldots+(2\mu'''_1+N'''_1)X'''_1}\ldots v_r^{(2\mu_r+N_r)X_r+\ldots+(2\mu'''_r+N'''_r)X'''_r}$$

$$P_{s,t}=(-1)^{\sum\limits_{t=1}^{t=r}(\mu_t\Lambda_t+\mu'_t\Lambda'_t+\mu''_t\Lambda''_t+\mu'''_t\Lambda'''_t)}\,p_1^{\frac{1}{4}\{(2\mu_1+N_1)^2+\ldots+(2\mu''_1+N'''_1)^2\}}\ldots p_t^{\frac{1}{4}\{(2\mu_t+1+N_t)^2+\ldots+(2\mu'''_t+1+N'''_t)^2\}}$$
$$p_s^{\frac{1}{4}\{(2\mu_s+1+N_s)^2+\ldots+(2\mu''_s+1+N'''_s)^2\}}\ldots p_r^{\frac{1}{4}\{(2\mu_r+N_r)^2+\ldots+(2\mu'''_r+N'''_r)^2\}}\ldots$$
$$p_{1,t}^{\frac{1}{2}\{(2\mu_1+N_1)(2\mu_t+1+N_t)+\ldots+(2\mu'''_1+N''_1)(2\mu'''_t+1+N'''_t)\}}p_{s,t}^{\frac{1}{2}\{(2\mu_s+1+N_s)(2\mu_t+1+N_t)+\ldots+(2\mu'''_s+1+N'''_s)(2\mu'''_t+1+N'''_t)\}}$$
$$v_1^{(2\mu_1+N_1)X_1+\ldots+(2\mu'''_1+N''_1)X'''_1}\ldots v_s^{(2\mu_s+1+N_s)X_s+\ldots+(2\mu'''_s+1+N'''_s)X'''_s}\ldots$$
$$v_t^{(2\mu_t+1+N_t)X_t+\ldots+(2\mu'''_t+1+N'''_t)X'''_t}\ldots v_r^{(2\mu_r+N_r)X_r+\ldots+(2\mu'''_r+N''_r)X'''_r}$$

and similarly for the others. Taking the terms in (228) separately we have

$$2^r\Sigma_0 P_0 = \Sigma_0 P_0 + \mathop{S}_{t=1}^{t=r} \Sigma_0(-1)^{\Sigma\mu_t} P_0 + \mathop{S}_{t=1}^{t=r}\mathop{S}_{s=1}^{s=r} \Sigma_0(-1)^{\Sigma\mu_t+\Sigma\mu_s} P_0$$
$$+ \mathop{S}_{t=1}^{t=r}\mathop{S}_{s=1}^{s=r}\mathop{S}_{u=1}^{u=r} \Sigma_0(-1)^{\Sigma\mu_t+\Sigma\mu_s+\Sigma\mu_u} P_0 + \ldots \quad (229)$$

in which

$$\Sigma\mu_t = \mu_t + \mu'_t + \mu''_t + \mu'''_t$$

and the summation on the right-hand side is taken for all values of the μ's, without restriction, from $-\infty$ to $+\infty$; the factor 2^r is prefixed to the left-hand side because there would remain 2^r terms of the initial value of $\Sigma_0 P_0$ were the right-hand side written out at full length. Now

$$P_0 = \text{general term in } \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s, \ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_s \ldots, X_r\right\}$$

$$(-1)^{\Sigma\mu_1} P_0 = \quad ,, \quad ,, \quad \Pi\Phi\left\{\begin{pmatrix}\Lambda_1+1, \Lambda_2, \ldots, \Lambda_s, \ldots, \Lambda_r\\ N_1\quad, N_2, \ldots, N_s, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_s, \ldots, X_r\right\}$$

. . . .

and therefore

$$2^r\Sigma_0 P_0 = \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s, \ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_r\right.$$
$$+ \sum_{s=1}^{s=r} \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s+1, \ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s\quad, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_r\right\}$$
$$+ \sum_{s=1}^{s=r}\sum_{t=1}^{t=r} \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s+1, \Lambda_t+1, \ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s\quad, N_t\quad, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_r\right\}$$
$$+ \sum_{s=1}^{s=r}\sum_{t=1}^{t=r}\sum_{u=1}^{u=r} \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s+1, \Lambda_t+1, \Lambda_u+1, \ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s\quad, N_t\quad, N_u\quad, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_r\right\}$$
$$+ \ldots \quad (230)$$

where in that expression on the right-hand side which has κ of the upper row of numbers in its characteristic of the form Λ_s+1 there will be $\frac{r!}{\kappa!\,\overline{r-\kappa}!}$ independent terms, and these of course are the only ones to be included in the "κ" tuple summation. Thus $2^r\Sigma_0 P_0$ is equal to the sum of 2^r products of four functions, to each product being prefixed a positive sign.

Again

$$2^r\Sigma_t P_t = \Sigma P_t - \Sigma(-1)^{\Sigma\mu_t} P_t + \mathop{S}_{s=1}^{s=r} \Sigma(-1)^{\Sigma\mu_s} P_t - \mathop{S}_{s=1}^{s=r} \Sigma(-1)^{\Sigma\mu_t+\Sigma\mu_s} P_t$$
$$+ \mathop{S}_{s=1}^{s=r}\mathop{S}_{u=1}^{u=r} \Sigma(-1)^{\Sigma\mu_s+\Sigma\mu_u} P_t - \ldots \quad (231)$$

where in the third and fourth terms s is not to take the value t; in the fifth term neither s nor u may take the value t, and so on; and all restrictions on the values of the μ's have been removed. In the expression the number of negative terms will be

$$1+r-1+\frac{r-1.r-2}{1.2}+\ldots+r-1+1=2^{r-1};$$

i.e., half the terms will be affected with a negative sign. Now

$$P_t=\text{general term in } \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_t+1, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}$$

$$(-1)^{\Sigma\mu_s}P_t= \quad ,, \quad ,, \quad \Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s+1, \ldots, \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s & ,\ldots, N_t+1, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}$$

and so for the others: thus

$$\begin{aligned}2\Sigma_tP_t=&\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_t+1, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}\\
&+\sum_{s=1}^{s=r}\pm\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s+1, \ldots, \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s & ,\ldots, N_t+1, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}\\
&+\sum_{s=1}^{s=r}\sum_{u=1}^{u=r}\pm\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s+1, \ldots, \Lambda_t & ,\ldots, \Lambda_u+1, \ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s & ,\ldots, N_t+1, \ldots N_u & ,\ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}\\
&+\ldots \qquad (232)\end{aligned}$$

an equal number (2^{r-1}) of terms being affected with a positive and with a negative sign; the rule by which it is applied will shortly be pointed out. Similarly

$$\begin{aligned}2^r\Sigma_{s,t}P_{s,t}=&\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_s & , \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_s+1, N_t+1, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}\\
&+\sum_{\kappa=1}^{\kappa=r}\pm\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_\kappa+1, \ldots, \Lambda_s & ,\ldots, \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_\kappa & ,\ldots, N_s+1, \ldots, N_t+1, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}\\
&+\sum_{\kappa=1}^{\kappa=r}\sum_{l=1}^{l=r}\pm\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_\kappa+1, \ldots, \Lambda_l+1, \ldots, \Lambda_s & ,\ldots, \Lambda_t & ,\ldots, \Lambda_r\\ N_1, N_2, \ldots, N_\kappa & ,\ldots, N_l & ,\ldots, N_s+1, \ldots, N_t+1, \ldots, N_r\end{pmatrix}\right.\\
&\left. X_1, X_2, \ldots, X_r\right\}+\ldots \qquad (233)\end{aligned}$$

half the terms, as in (232), being affected with a negative sign.

Proceeding in this manner, (228) gives

$$2^r\Phi\Phi'\Phi''\Phi'''=\text{sum of } 4^r \text{ products of four functions } \Phi \quad \ldots \quad (234).$$

63. Each of these products is affected with a positive or negative sign, determined by the following rule (Rule I.); and it has this sign modified by being multiplied, according to another rule (Rule II.), by a definite power of negative unity. Taking the first term, viz.:—

$$\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_r \\ N_1, N_2, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\}$$

the characteristics for the remaining 4^r-1 similar products may be written down as follows: for the upper rows take all possible combinations of

$$\begin{cases}\Lambda_1, \\ \Lambda_1+1,\end{cases}\begin{cases}\Lambda_2, \\ \Lambda_2+1,\end{cases}\begin{cases}\Lambda_3, \\ \Lambda_3+1,\end{cases}\ldots,\begin{cases}\Lambda_s, \\ \Lambda_s+1,\end{cases}\ldots,\begin{cases}\Lambda_r, \\ \Lambda_r+1,\end{cases}$$

by selecting one from each bracketed pair; and similarly, for the lower rows, from

$$\begin{cases}N_1, \\ N_1+1,\end{cases}\begin{cases}N_2, \\ N_2+1,\end{cases}\begin{cases}N_3, \\ N_3+1,\end{cases}\ldots,\begin{cases}N_s, \\ N_s+1,\end{cases}\ldots,\begin{cases}N_r, \\ N_r+1.\end{cases}$$

Defining that number in the lower row of the characteristic as "corresponding" with another in the upper row (and vice versâ) when the two have the same suffix, the following are the rules above referred to:—

Rule I. If, in the typical characteristic of any product, there be an odd number of pairs of corresponding numbers such that each member of a pair differs by unity from the member holding the same position in the first term, viz.: in

$$\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_r \\ N_1, N_2, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\},$$

then to that product is prefixed a negative sign; but if there be an even number of such pairs, a positive sign must be prefixed.

(As in the algebraical expression of the theorem the numbers will be of the form N_s+1, N_t+1 and not of the form $N_s-1=N_s+1-2$, $N_t-1=N_t+1-2$, which might by formula (220) cause a difference of sign, the above rule is perfectly determinate).

Rule II. To find the index of $(-1)^{\frac{1}{2}}$ in order to prefix the proper power of (-1) to a product, there must be taken the sum of the numbers in the upper rows of the characteristics of the four functions in the first term, viz.: in

$$\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \ldots, \Lambda_r \\ N_1, N_2, \ldots, N_r\end{pmatrix}X_1, X_2, \ldots, X_r\right\},$$

corresponding to those in the lower row which hold the same position as the numbers, differing from them by unity, in the lower row of the typical characteristic of the product.

(Since it has been assumed that

$$\Lambda_t+\Lambda'_t+\Lambda''_t+\Lambda'''_t=\text{even}$$

for all values of t which occur, no imaginary quantities are introduced).

Thus, as a term on the right-hand side, there will be

$$-(-1)^{\frac{1}{2}[\Sigma\Lambda_l+\Sigma\Lambda_m+\Sigma\Lambda_n+\Sigma\Lambda_p]}$$

$$\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \Lambda_3+1, \ldots, \Lambda_\kappa+1, \Lambda_l+1, \Lambda_m+1, \Lambda_n+1, \Lambda_p \quad, \Lambda_q, \ldots, \Lambda_r \\ N_1, N_2, N_3 \quad, \ldots, N_\kappa \quad, N_l+1, N_m+1, N_n+1, N_p+1, N_q, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_r\right\}$$

where

$$\Sigma\Lambda_l=\Lambda_l+\Lambda'_l+\Lambda''_l+\Lambda'''_l.$$

The coefficient of $\frac{1}{2}$ in the index of -1 is $\Sigma\Lambda_l+\Sigma\Lambda_m+\Sigma\Lambda_n+\Sigma\Lambda_p$, by Rule II., since the numbers N_l+1, N_m+1, N_n+1, N_p+1 are all that differ by unity from those in the first term, and the sum of the numbers which correspond to $N_l, \ldots, N_m, \ldots, N_n, \ldots, N_p, \ldots$, in the four functions in that term is $\Sigma\Lambda_l+\Sigma\Lambda_m+\Sigma\Lambda_n+\Sigma\Lambda_p$; and a $-$ sign is prefixed, by Rule I., because there is an odd number of pairs of corresponding numbers—$\begin{cases}\Lambda_l+1, \\ N_l+1,\end{cases}\begin{cases}\Lambda_m+1, \\ N_m+1,\end{cases}\begin{cases}\Lambda_n+1, \\ N_n+1,\end{cases}$ each member of which differs by unity from the members of the similarly situated pairs in the first term. So another term will be

$$(-1)^{\frac{1}{2}\{\Sigma\Lambda_l+\Sigma\Lambda_m+\Sigma\Lambda_n+\Sigma\Lambda_q\}}\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \Lambda_3+1, \ldots, \Lambda_k+1, \Lambda_l+1, \\ N_1, N_2, N_3 \quad, \ldots, N_k \quad, N_l+1,\end{pmatrix}\right.$$

$$\left.\begin{matrix}\Lambda_m+1, \Lambda_n \quad, \ldots, \Lambda_p+1, \Lambda_q \quad, \ldots, \Lambda_r \\ N_m+1, N_n+1, \ldots, N_p \quad, N_q+1, \ldots, N_r\end{matrix}\Bigg) X_1, X_2, \ldots, X_r\right\};$$

and the sign and coefficient of any term may be written down from an inspection of its characteristic.

64. As it has been proved that every number in the characteristic is either zero or unity, and the assumption has been made that the sum of any four similarly situated numbers in the characteristics of the four functions is even, the general product theorem comprises $(4^r)^3$, *i.e.*, 2^{6r}, particular cases, the variables being still left perfectly general.

65. In a manner similar to that adopted in Section I. the following formulæ are obtainable :—

$$\left.\begin{array}{l}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix}+\overset{t=r}{\underset{t=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_t+1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix}\\ \quad+\overset{s=r}{\underset{s=1}{\mathrm{S}}}\overset{t=r}{\underset{t=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \ldots, \lambda_t+1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_s \quad, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix}+\ldots\\ \quad\ldots+\overset{s=r}{\underset{s=1}{\mathrm{S}}}\overset{t=r}{\underset{t=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \lambda_3+1, \ldots, \lambda_s, \ldots, \lambda_t, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \nu_3 \quad, \ldots, \nu_s, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix}\\ \quad+\overset{s=r}{\underset{s=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix}+\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_r\end{pmatrix}\\ =\text{same expression with } \begin{matrix}\Lambda \text{ written for } \lambda\\ \mathrm{N} \quad ,, \quad \nu\end{matrix}\Big\}\text{throughout}\end{array}\right\} \quad . \quad (235)$$

the variables on the left-hand side being

$$x_1, x_2, \ldots x_r, x'_1, x'_2, \ldots x'_r, x''_1, x''_2, \ldots x''_r, x'''_1, x'''_2, \ldots x'''_r,$$

and the variables on the right-hand side being

$$\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_r, \mathrm{X}'_1, \mathrm{X}'_2, \ldots, \mathrm{X}'_r, \mathrm{X}''_1, \mathrm{X}''_2, \ldots, \mathrm{X}''_r, \mathrm{X}'''_1, \mathrm{X}'''_2, \ldots, \mathrm{X}'''_r.$$

Again

$$\left.\begin{array}{l}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_t+1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix}\\ \quad+\overset{s=r}{\underset{s=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix}\\ \quad-\overset{s=r}{\underset{s=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \ldots, \lambda_t+1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_s \quad, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix}\\ \quad+\overset{s=r}{\underset{s=1}{\mathrm{S}}}\overset{u=r}{\underset{u=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \ldots, \lambda_u+1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_s \quad, \ldots, \nu_u \quad, \ldots, \nu_r\end{pmatrix}-\ldots\\ \quad\ldots-\overset{s=r}{\underset{s=1}{\mathrm{S}}}\overset{u=r}{\underset{u=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s, \ldots, \lambda_u, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s, \ldots, \nu_u, \ldots, \nu_r\end{pmatrix}\\ \quad+\overset{s=r}{\underset{s=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s, \ldots, \lambda_t, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix}\\ \quad-\overset{s=r}{\underset{s=1}{\mathrm{S}}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s, \ldots, \nu_r\end{pmatrix}\\ \quad+\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_t, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix}-\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_r+1\\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_r\end{pmatrix}\\ =(-1)^{\frac{1}{2}\{\lambda_t+\lambda'_t+\lambda''_t+\lambda'''_t\}}[\text{same expression with } \Lambda \text{ written for } \lambda \text{ and N for } \nu\\ \qquad\qquad \text{throughout except for } \nu_t, \text{ for which write } \mathrm{N}_t+1\,;\\ \qquad\qquad \text{the variables being as in (235)}]\end{array}\right\} \quad . \quad (236)$$

This equation (236) is equivalent to r equations by substituting in succession $t=1, 2, 3, \ldots, r$; and a summation indicated by $\overset{s=r}{\underset{s=1}{\mathrm{S}}}$ implies that s is to have every integer value $1, 2, 3, \ldots, r$, except t. Again

$$
\left.
\begin{aligned}
&\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix} - \Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \ldots, \lambda_t, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s \quad, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix} \\
&\qquad -\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_t+1, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&\qquad +\mathop{S}_{l=1}^{l=r}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_l+1, \ldots, \lambda_s, \lambda_t, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_l \quad, \ldots, \nu_s, \nu_t, \ldots, \nu_r\end{pmatrix} \\
&\qquad +\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s+1, \lambda_t+1. \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s \quad, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&\qquad -\mathop{S}_{l=1}^{l=r}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_l+1, \ldots, \lambda_s, \ldots, \lambda_t+1, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_l \quad, \ldots, \nu_s, \ldots, \nu_t \quad, \ldots. \nu_r\end{pmatrix} \\
&\qquad -\mathop{S}_{l=1}^{l=r}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_l+1, \ldots, \lambda_s+1, \ldots, \lambda_t, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_l \quad, \ldots, \nu_s \quad, \ldots, \nu_t, \ldots. \nu_r\end{pmatrix} \\
&\qquad +\mathop{S}_{l=1}^{l=r}\mathop{S}_{m=1}^{m=r}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_l+1, \ldots, \lambda_m+1, \ldots, \lambda_s, \ldots, \lambda_t, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_l \quad, \ldots, \nu_m \quad, \ldots, \nu_s, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix} - \ldots \\
&\quad \ldots +\mathop{S}_{l=1}^{l=r}\mathop{S}_{m=1}^{m=r}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_l, \ldots, \lambda_m, \ldots, \lambda_s+1, \ldots, \lambda_t+1, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_l, \ldots, \nu_m, \ldots, \nu_s \quad, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&\qquad -\mathop{S}_{l=r}^{l=r}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_l, \ldots, \lambda_s, \lambda_t+1, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_l, \ldots, \nu_s, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&\qquad -\mathop{S}_{l=1}^{l=r}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_l, \ldots, \lambda_s+1, \ldots, \lambda_t, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_l, \ldots, \nu_s \quad, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix} \\
&\qquad +\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s, \ldots, \lambda_t, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s, \ldots, \nu_t, \ldots. \nu_r\end{pmatrix} \\
&\qquad +\mathop{S}_{l=1}^{l=r}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_l, \ldots, \lambda_s+1, \ldots, \lambda_t+1, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_l, \ldots, \nu_s \quad, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&\qquad -\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s, \ldots, \lambda_t+1, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&\qquad -\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1. \ldots, \lambda_s+1, \ldots, \lambda_t, \ldots, \lambda_r+1 \\ \nu_1 \quad, \nu_2 \quad, \ldots, \nu_s \quad, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix} \\
&\qquad +\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s+1, \ldots, \lambda_t+1, \ldots, \lambda_r+1 \\ \nu_1 \quad. \nu_2 \quad, \ldots, \nu_s \quad, \ldots, \nu_t \quad, \ldots, \nu_r\end{pmatrix} \\
&= (-1)^{\frac{1}{2}(\lambda_t+\lambda'_t+\lambda''_t+\lambda'''_t+\lambda_s+\lambda'_s+\lambda''_s+\lambda'''_s)} \text{[same expression with } \Lambda \text{ written for } \lambda \text{ and N} \\
&\qquad\qquad \text{for } \nu \text{ throughout, except for } \nu_s, \nu_t \text{ for} \\
&\qquad\qquad \text{which write respectively } N_s+1, N_t+1\,; \\
&\qquad\qquad \text{the variables being as in (235)]}
\end{aligned}
\right\} \quad (237)
$$

in which s, t are to have in succession the values 1, 2, 3, . . . , r, (but never the same value together); and $\mathop{S}\limits_{l=1}^{l=r}$ implies summation for every value of l in 1, 2, 3, . . . , r, except $l=s$, and $l=t$. Thus the equation comprises $\frac{1}{2}r(r-1)$ cases.

66. From an inspection of these equations, it is seen that the lower row in the characteristic of each term is the same throughout the same side of the same equation; and this holds throughout the system of 2^r equations of which the above are examples. To write down the equation in which κ of the numbers in the lower row on the right-hand side differ by unity from $N_1, N_2, \ldots, N_r$, (and which is therefore an equation comprising $\frac{r!}{\kappa!\,r-\kappa!}$ cases), take that group of 2^r terms in the general product theorem having this lower row for the common lower row of the characteristic and multiply the group by

$$(-1)^{\frac{1}{2}\overset{t}{S}\Sigma\Lambda_t}$$

where $\Sigma\Lambda_t$ has already been defined and $\overset{t}{S}$ implies that summation is to be taken for those κ values of t which have their numbers in the lower row of the characteristic of the form N_t+1; this is the right-hand side of the equation. To obtain the left-hand side the coefficient $(-1)^{\frac{1}{2}\overset{t}{S}\Sigma\Lambda_t}$ is dropped, as well as all the units in the numbers $N_t+1, \ldots$; and λ, ν, x are substituted for Λ, N, X respectively. Thus to a term of the form

$$(-1)^{\frac{1}{2}(\Sigma\Lambda_l+\Sigma\Lambda_m+\Sigma\Lambda_n+\Sigma\Lambda_q)}\Pi\Phi\left\{\begin{pmatrix}\Lambda_1, \Lambda_2, \Lambda_3+1, \ldots, \Lambda_\kappa+1, \ldots, \Lambda_l+1, \ldots, \Lambda_m+1, \ldots, \Lambda_n, \ldots, \Lambda_p+1, \ldots, \Lambda_q, \ldots, \Lambda_r \\ N_1, N_2, N_3, \ldots, N_\kappa, \ldots, N_l+1, \ldots, N_m+1, \ldots, N_n+1, \ldots, N_p, \ldots, N_q+1, \ldots, N_r\end{pmatrix} X_1, X_2, \ldots, X_r\right\}$$

on the right-hand side, there will on the left-hand side be a term of the form

$$\Pi\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \lambda_3+1, \ldots, \lambda_\kappa+1, \ldots, \lambda_l+1, \ldots, \lambda_m+1, \ldots, \lambda_n, \ldots, \lambda_p+1, \ldots, \lambda_q, \ldots, \lambda_r \\ \nu_1, \nu_2, \nu_3, \ldots, \nu_\kappa, \ldots, \nu_l, \ldots, \nu_m, \ldots, \nu_n, \ldots, \nu_p, \ldots, \nu_q, \ldots, \nu_r\end{pmatrix} x_1, x_2, \ldots, x_r\right\}$$

67. By increasing each variable x (and therefore also X, from its definition) by the quarter quasi-period in any set of conjugate quasi-periods, and taking all combinations of these (amounting in number to 2^r), each equation of the above system of 2^r equations gives rise to 2^r further equations. The reason of this is that each function is periodic with the exception of a factor

$$p_s^{-\frac{1}{4}}e^{-\frac{i\pi x_s}{2K_s}}$$

and therefore a product on the left-hand side is periodic with the exception of a factor

$$p_s^{-1}e^{-\frac{i\pi}{2K_s}(x_s+x'_s+x''_s+x'''_s)}$$

while the corresponding factor on the right-hand side is

$$p_s^{-1}e^{-\frac{i\pi}{2K_s}(X_s+X'_s+X''_s+X'''_s)}$$

and these are equal in virtue of the relation

$$x_s+x'_s+x''_s+x'''_s=X_s+X'_s+X''_s+X'''_s.$$

As an example, by the substitutions $x_1+\frac{K_1}{\pi i}\log p_{1,s}$ for $x_1, \ldots, x_s+\frac{K_s}{\pi i}\log p_s$ for $x_s, \ldots,$ (235) gives

$$\left.\begin{array}{l}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s+1, \ldots, \nu_r\end{pmatrix}+\overset{t=r}{\underset{t=1}{S}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_t+1, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s+1, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix} \\ \quad +\overset{t=r}{\underset{t=1}{S}}\overset{u=r}{\underset{u=1}{S}}\Pi\Phi\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_t+1, \ldots, \lambda_u+1, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_s+1, \ldots, \nu_t, \ldots, \nu_u, \ldots, \nu_r\end{pmatrix}+\ldots \\ \quad +\overset{t=r}{\underset{t=1}{S}}\overset{u=r}{\underset{u=1}{S}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s+1, \lambda_t, \lambda_u, \ldots, \lambda_r+1 \\ \nu_1, \nu_2, \ldots, \nu_s+1, \nu_t, \nu_u, \ldots, \nu_r\end{pmatrix} \\ \quad +\overset{t=r}{\underset{t=1}{S}}\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s+1, \ldots, \lambda_t, \ldots, \lambda_r+1 \\ \nu_1, \nu_2, \ldots, \nu_s+1, \ldots, \nu_t, \ldots, \nu_r\end{pmatrix} \\ \quad +\Pi\Phi\begin{pmatrix}\lambda_1+1, \lambda_2+1, \ldots, \lambda_s+1, \ldots, \lambda_r+1 \\ \nu_1, \nu_2, \ldots, \nu_s+1, \ldots, \nu_r\end{pmatrix} \\ = \text{same expression with } \Lambda, \text{N}, \text{X written throughout for } \lambda, \nu, x \text{ respectively}\end{array}\right\} \quad (238).$$

The same remark with regard to (238) may be made here as at the end of § 8.

68. We obviously have from the definition

$$\Phi_r\left[\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix}x_1, x_2, \ldots, x_r\right]$$

$$=\sum_{m_1=-\infty}^{m_1=\infty}(-1)^{m_1\lambda_1}p_1^{\left(m_1+\frac{\nu_1}{2}\right)^2}e^{(2m_1+\nu_1)\frac{i\pi x_1}{2K_1}}\Phi_{r-1}\left[\begin{pmatrix}\lambda_2, \lambda_3, \ldots, \lambda_r \\ \nu_2, \nu_3, \ldots, \nu_r\end{pmatrix}x'_2, x'_3, \ldots, x'_r\right]. \quad (239)$$

where Φ_r, Φ_{r-1} are functions of the orders r, $r-1$ respectively, and

$$\left.\begin{array}{l}x'_s=x_s+\frac{2m_1+\nu_1}{2}\log p'_{1,s} \\ \log p_{1,s}=\frac{i\pi}{2K_s}\log p'_{1,s}\end{array}\right\} \quad \ldots\ldots\ldots \quad (240).$$

Putting all the numbers in the characteristic of Φ_r zero, we have

$$\underset{0,r}{\Phi}(x_1, x_2, \ldots, x_r)$$

$$= \sum_{m_1=-\infty}^{m_1=\infty} p_1^{m_1^2} e^{m_1 \frac{i\pi x_1}{K_1}} \underset{0,r-1}{\Phi}(x_2+m_1 \log p'_{1,2}, x_3+m_1 \log p'_{1,3}, \ldots, x_r+m_1 \log p'_{1,r})$$

$$= \underset{0,r-1}{\Phi}(x_2, x_3, \ldots, x_r)+p_1 \cos \frac{\pi x_1}{K_1}\left[\underset{0,r-1}{\Phi}(x_2+\log p'_{1,2}, \ldots)+\underset{0,r-1}{\Phi}(x_2-\log p'_{1,2}, \ldots)\right]$$
$$+ip_1 \sin \frac{\pi x_1}{K_1}\left[\underset{0,r-1}{\Phi}(x_2+\log p'_{1,2}, \ldots)-\underset{0,r-1}{\Phi}(x_2-\log p'_{1,2}, \ldots)\right]$$
$$+p_1^4 \cos \frac{2\pi x_1}{K_1}\left[\underset{0,r-1}{\Phi}(x_2+2\log p'_{1,2}, \ldots)+\underset{0,r-1}{\Phi}(x_2-2\log p'_{1,2}, \ldots)\right]$$
$$+ip_1^4 \sin \frac{2\pi x_1}{K_1}\left[\underset{0,r-1}{\Phi}(x_2+2\log p'_{1,2}, \ldots)-\underset{0,r-1}{\Phi}(x_2-2\log p'_{1,2}, \ldots)\right]$$
$$+ \ldots$$

Expanding and arranging, this gives

$$\underset{0,r}{\Phi}(x_1, x_2, \ldots, x_r)=\underset{0,r-1}{\Phi}(x_2, x_3, \ldots, x_r)\left[1+2p_1 \cos \frac{\pi x_1}{K_1}+2p_1^4 \cos \frac{2\pi x_1}{K_1}+ \ldots\right]$$
$$+2i\left\{p_1 \sin \frac{\pi x_1}{K_1}+2p_1^4 \sin \frac{2\pi x_1}{K_1}+3p_1^9 \sin \frac{3\pi x_1}{K_1}+ \ldots\right\}\left(\log p'_{1,2}\frac{d}{dx_2}+\log p'_{1,3}\frac{d}{dx_3}+\ldots\right)\underset{0,r-1}{\Phi}$$
$$+\frac{1}{2!}\, 2\left\{p_1 \cos \frac{\pi x_1}{K_1}+2^2.p_1^4 \cos \frac{2\pi x_1}{K_1}+ \ldots\right\}\left(\log p'_{1,2}\frac{d}{dx_2}+\log p'_{1,3}\frac{d}{dx_3}+ \ldots\right)^2 \underset{0,r-1}{\Phi}$$
$$+ \ldots$$
$$= \underset{0,r-1}{\Phi}\,\theta_{0,0}(x_1)-\frac{K_1}{\pi}\frac{d\theta_{0,0}(x_1)}{dx_1}\left[\frac{2K_2}{\pi}\log p_{1,2}\frac{d}{dx_2}+\frac{2K_3}{\pi}\log p_{1,3}\frac{d}{dx_3}+ \ldots\right]\underset{0,r-1}{\Phi}$$
$$+\frac{1}{2!}\left(\frac{K_1}{\pi}\right)^2 \frac{d^2\theta_{0,0}(x_1)}{dx_1^2}\left[\frac{2K_2}{\pi}\log p_{1,2}\frac{d}{dx_2}+\frac{2K_3}{\pi}\log p_{1,3}\frac{d}{dx_3}+ \ldots\right]\underset{0,r-1}{\Phi}+ \ldots \quad (241)$$

$$= e^{-\frac{2K_1}{\pi^2}\sum\limits_{s=2}^{s=r}(\log p_{1,s})K_s\frac{d^2}{dx_1 dx_s}}\theta_{0,0}(x_1)\underset{0,r-1}{\Phi}(x_2, x_3, \ldots, x_r) \quad . \quad . \quad . \quad . \quad . \quad (242)$$

$$= e^{-\frac{2}{\pi^2}\sum\limits_{s=1}^{s=r}\sum\limits_{t=1}^{t=r} K_s K_t \log p_{s,t}\frac{d^2}{dx_s dx_t}}\prod_{t=1}^{t=r}\theta_{0,0}(x_t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (243),$$

where the double summation in the index implies that s, t are to have every value 1, 2, 3. . . . , r (but never the same value together); and

$$\prod_{t=1}^{t=r}\theta_{0,0}(x_t)=\theta_{0,0}(x_1)\theta_{0,0}(x_2) \ldots \theta_{0,0}(x_r).$$

The theorem for the general function is

$$\Phi\left\{\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, \nu_r\end{pmatrix}x_1, x_2, \ldots, x_r\right\}=e^{-\frac{2}{\pi^2}\sum\limits_{s=1}^{s=r}\sum\limits_{t=1}^{t=r} K_s K_t \log p_{s,t}\frac{d^2}{dx_s dx_t}}\prod_{u=1}^{u=r}\theta_{\nu_u, \lambda_u}(x_u) \quad . \quad (244).$$

69. Since $\theta(x_u)$ satisfies the general differential equation

$$\frac{d^2\theta}{dx_u^2}-2x_u\left(\kappa'^2_u-\frac{\mathrm{E}_u}{\mathrm{K}_u}\right)\frac{d\theta}{dx_u}+2\kappa_u\kappa'^2_u\frac{d\theta}{d\kappa_u}=0$$

and the general term in Φ, so far as concerns x_u, is a numerical multiple of

$$\mathrm{K}_u^m\frac{d^m\theta(x_u)}{dx_u^m}$$

it follows, exactly as in Section II., that Φ satisfies the r equations of the form

$$\frac{d^2\Phi}{dx_u^2}-2x_u\left(\kappa'^2_u-\frac{\mathrm{E}_u}{\mathrm{K}_u}\right)\frac{d\Phi}{dx_u}+2\kappa_u\kappa'^2_u\frac{d\Phi}{d\kappa_u}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (245).$$

That this is satisfied can be verified by means of the definition of Φ; and the same is true of the $\frac{1}{2}r(r-1)$ equations of the type

$$p_{s,t}\frac{d\Phi}{dp_{s,t}}+\frac{2\mathrm{K}_s\mathrm{K}_t}{\pi^2}\frac{d^2\Phi}{dx_s dx_t}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (246)$$

all satisfied by Φ.

70. Expressions for the constant terms in the even functions and for all coefficients in the expansions of all the functions in powers of the x's may be obtained as before. Noticing that

$$\theta_{\nu,\lambda}(0)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\mathrm{K}^{\frac{1}{2}}c^{\frac{\nu}{4}}c'^{\frac{\lambda}{4}}$$

(ν, λ being either zero or unity, but not both unity at the same time) we have

$$\mathrm{C}\begin{pmatrix}\lambda_1, & \lambda_2, & \ldots, & \lambda_r\\ \nu_1, & \nu_2, & \ldots, & \nu_r\end{pmatrix}=\left(\frac{2}{\pi}\right)^{\frac{r}{2}}\nabla_0\cdot\prod_{t=1}^{t=r}\mathrm{K}_t^{\frac{1}{2}}c_t^{\frac{\nu_t}{4}}c'^{\frac{\lambda_t}{4}}_t \quad . \quad . \quad . \quad . \quad . \quad . \quad (247)$$

where

$$\nabla_0=\cosh\left\{2\sum_{s=1}^{s=r}\sum_{t=1}^{t=r}\log p_{s,t}\left(\frac{d^2}{dp_s dp_t}\right)^{\frac{1}{2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (248)$$

and in the summation s, t are not to have the same value together. This gives the constant term in all those 3^r even functions in the characteristics of which no two corresponding numbers are unity at the same time. Similarly if we put

$$\pi^2\nabla_{l,m}=2\log p_{l,m}+\frac{2^3}{3!}\left\{(\log p_{l,m})^3\frac{d^2}{dp_ldp_m}+3\log p_{l,m}\sum_{s=1}^{s=r}\sum_{t=1}^{t=r}(\log p_{s,t})^2\frac{d^2}{dp_sdp_t}\right\}+\ldots$$

$$=\frac{1}{2\left(\frac{d^2}{dp_ldp_m}\right)^{\frac{1}{2}}}\left[\sinh 2\left\{\log p_{l,m}\left(\frac{d^2}{dp_ldp_m}\right)^{\frac{1}{2}}+\sum_{s=1}^{s=r}\sum_{t=1}^{t=r}\log p_{s,t}\left(\frac{d^2}{dp_sdp_t}\right)^{\frac{1}{2}}\right\}\right.$$

$$\left.+\sinh 2\left\{\log p_{l,m}\left(\frac{d^2}{dp_ldp_m}\right)-\sum_{s=1}^{s=r}\sum_{t=1}^{t=r}\log p_{s,t}\left(\frac{d^2}{dp_sdp_t}\right)^{\frac{1}{2}}\right\}\right] \quad . \quad . \quad . \quad (249)$$

where in the summation s, t take all the values 1, 2, 3, . . . , r, except l and m together, then

$$C\begin{pmatrix}\lambda_1, \lambda_2, \ldots, 1, \ldots, 1, \ldots, \lambda_r\\ \nu_1, \nu_2, \ldots, 1, \ldots, 1, \ldots, \nu_r\end{pmatrix}=\left(\frac{2}{\pi}\right)^{\frac{r}{2}}\nabla_{l,m}.c_l^{\frac{1}{4}}c'^{\frac{1}{4}}_l c_m^{\frac{1}{4}}c'^{\frac{1}{4}}_m K_l^{\frac{3}{2}}K_m^{\frac{3}{2}}\prod_{t=1}^{t=r}c_t^{\frac{\nu_t}{4}}c'^{\frac{\lambda_t}{4}}_t K_t^{\frac{1}{2}} \quad . \quad (250)$$

in which t has all values except l, m. This formula gives the constant terms for all those 3^{r-2} even functions in the characteristics of which $\lambda_l=\lambda_m=1=\nu_l=\nu_m$, but no other corresponding numbers are unity at the same time; and since l, m may be any whatever of the suffixes, this formula comprises $\frac{1}{2}r(r-1)3^{r-2}$ constants.

The above will suffice to indicate how all the constants may be obtained.

NOTE.—Since the above memoir was written I have seen a paper by Professor H. J. S. SMITH (in vol. x. of Lond. Math. Soc. Proc., 1879) in which the results of §§ 6, 62, and 63 are given in an equivalent but somewhat different and more concise form. [Sept. 29, 1882.]